

The CIA Model

Ryoichi Imai

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Time is continuous. There is no population growth. The economy produces output $f(k)$ where k is the per capita capital. Let c and i denote consumption and physical investment. There are two ways for the household to accumulate assets: investing in physical capital and holding money. Money is needed for both consumption and investment. The household is required to hold cash which covers consumption expense, while a fraction φ of investment is "cash goods". The price level is denoted by P . In this note, we eliminate time subscripts of each variable.

The budget constraint of a representative household is

$$\frac{\dot{M}}{P} = f(k) - c - \left(i + \frac{i^2}{2}\eta \right) + \frac{\tau}{P}$$

where τ denotes the nominal money transfer from the central bank to the household. We can rewrite this expression as a transitional equation of real money balance $m = M/P$.

$$\dot{m} = f(k) - c - \left(i + \frac{i^2}{2}\eta \right) + \frac{\tau}{P} - m\pi \quad (1)$$

where $\pi = \dot{P}/P$. The physical capital grows according to

$$\dot{k} = i - \delta k \quad (2)$$

where δ denotes the depreciation rate. The cash-in-advance constraint is written as

$$m \geq c + \varphi \left(i + \frac{i^2}{2}\eta \right) \quad (3)$$

The household maximizes its intertemporal utility,

$$\int_0^{\infty} u(c)e^{-rt} dt$$

subject to (1), (2), and (3). Then the current-value Hamiltonian function is written as

$$H = u(c) + \lambda \left[f(k) - c - \left(i + \frac{i^2}{2}\eta \right) + \frac{\tau}{P} - m\pi \right] + \mu(i - \delta k) + \nu \left[m - c - \varphi \left(i + \frac{i^2}{2}\eta \right) \right]$$

The Maximum Principle is obtained as

1. The FOCs:

$$H_c = u'(c) - (\lambda + \nu) = 0 \quad (4)$$

$$H_i = -\lambda - i\eta\lambda + \mu - \nu\varphi(1 + \eta) = 0 \quad (5)$$

2. The Hamiltonian Dynamics:

$$\dot{\lambda} = r\lambda - H_m = r\lambda + \pi\lambda - \nu \quad (6)$$

$$\dot{\mu} = r\mu - H_k = r\mu - \lambda f'(k) + \delta\mu \quad (7)$$

The equilibrium consists of $(c, i, m, k, \lambda, \mu, \nu)$ that satisfy (1) through (7). We focus on the steady state, in which $\dot{m} = \dot{k} = \dot{\lambda} = \dot{\mu} = 0$.

Assume that the CIA constraint is binding, that is, $\nu > 0$. Then, from (5) and (6), we have

$$\nu = \frac{-\lambda - i\eta\lambda + \mu}{\varphi(1 + \eta)} = \lambda(r + \pi)$$

which solves for

$$\mu = [\varphi(1 + \eta)(r + \pi) + (1 + i\eta)]\lambda \quad (8)$$

From (7) and (8), it follows that

$$f'(k) = \frac{\mu}{\lambda}(r + \delta) = (r + \delta)[\varphi(1 + \eta)(r + \pi) + (1 + i\eta)]$$

Using (2), we can rewrite the as expression as

$$f'(k) = \frac{\mu}{\lambda}(r + \delta) = (r + \delta)[\varphi(1 + \eta)(r + \pi) + (1 + \delta\eta k)] \quad (9)$$

The goods market clears if

$$f(k) - c - \left(i + \frac{i^2}{2}\eta\right) = f(k) - c - \left(\delta k + \frac{\delta^2 k^2}{2}\eta\right) = 0 \quad (10)$$

The money transfer follows $\tau = \sigma M$. Then the goods market clearing condition (10) and the steady state ($\dot{m} = 0$) imply

$$\frac{\tau}{P} - \pi m = \frac{\sigma M}{P} = (\sigma - \pi)m = 0$$

which results in that $\pi = \sigma$. Then we can rewrite (9) as

$$f'(k) = \frac{\mu}{\lambda}(r + \delta) = (r + \delta)[\varphi(1 + \eta)(r + \sigma) + (1 + \delta\eta k)] \quad (11)$$

This expression fully determines the steady state capital when the CIA constraint is binding, which is illustrated in Figure 1. There are two remarkable results. First, the nominal money affects the steady state capital only if the fraction of investment that is subject to CIA constraint is positive, $\varphi > 0$. Second, the higher nominal money growth (a larger σ) reduces the steady state capital.

Next, assume alternatively that the CIA constraint is not binding, $\nu = 0$. Then (6) implies $r + \pi = 0$. Also, (5) and (7) lead to $f'(k) = (r + \delta)(1 + \delta\eta k)$. In a competitive economy, the real interest rate is given by $f'(k)$. If we neglect the depreciation, that is, if $\delta = 0$, the real interest rate equals the discount rate, that is, $f'(k) = r$. Then, if the CIA constraint is not binding and the capital depreciation rate is negligibly small, the nominal interest rate equals zero; $f'(k) + \pi = 0$.

No matter if the CIA constraint is binding or not binding, the steady state capital clears the goods market as (10). Then the intertemporal utility is can be written as

$$\int_0^\infty u \left[f(k) - \left(\delta k + \frac{\delta^2 k^2}{2} \eta \right) \right] e^{-rt} dt$$

It is obvious that the intertemporal utility is maximized at k that satisfies

$$f'(k) = \delta(1 + \delta\eta k). \quad (12)$$

As is illustrated in Figure 2, the steady state capital with which the CIA constraint is not binding is closer to its social optimum level determined by (12), than the steady state capital with which the CIA constraint is binding. Therefore we can conclude that the zero nominal interest rate is optimal.

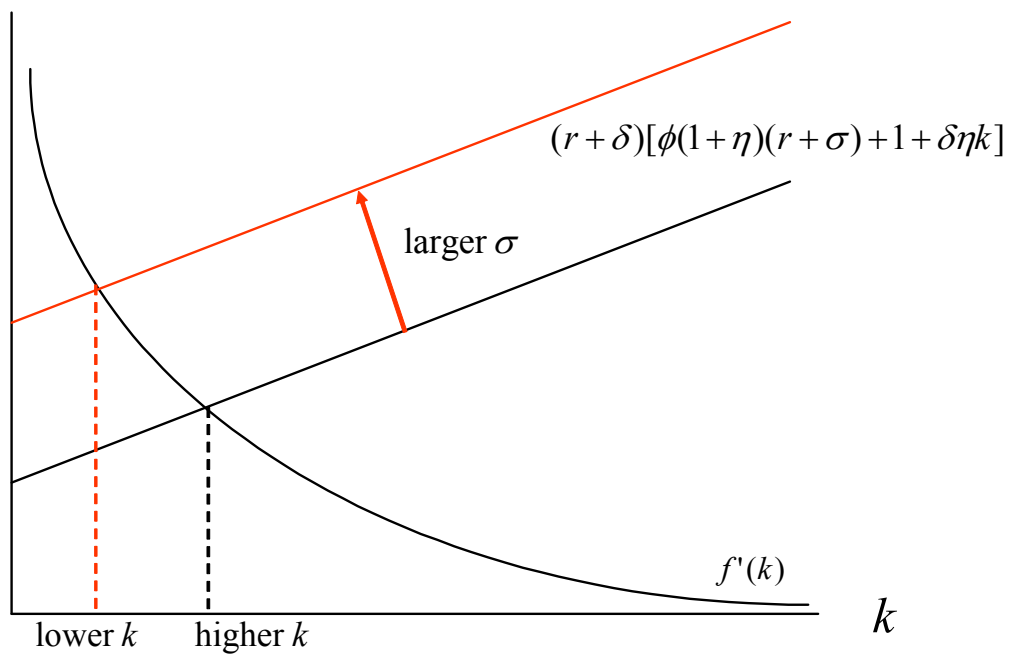


Figure 1:

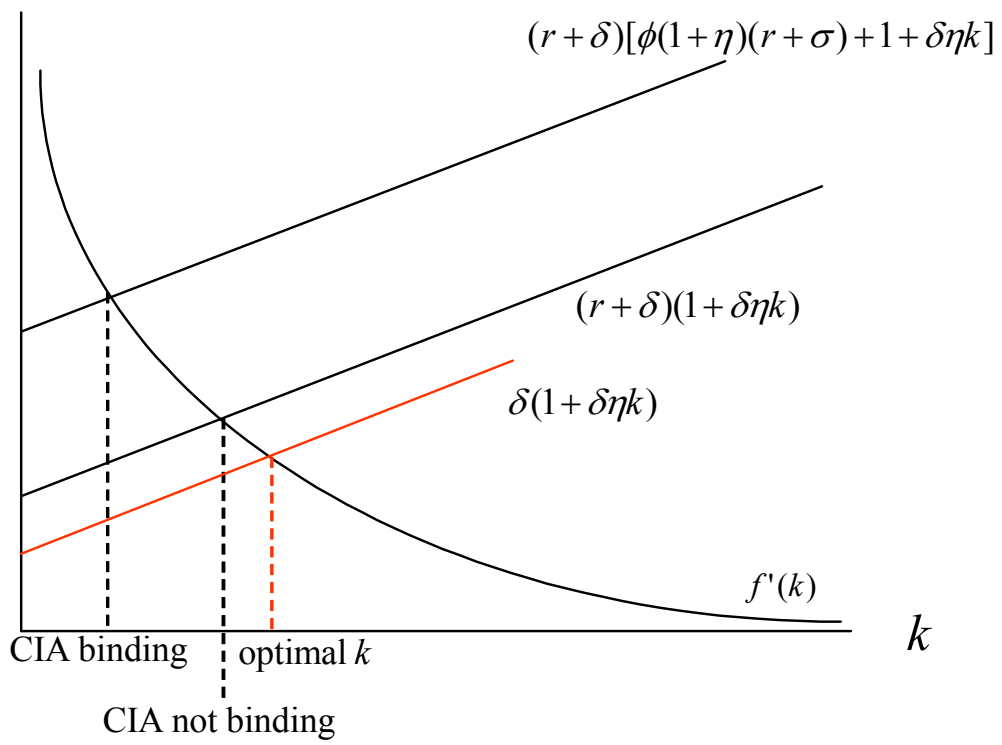


Figure 2: