

A Simple Model of Economic Consequences of the COVID-19 Pandemic

Ryoichi Imai

Kyushu University

August 29, 2020

The SIR (Susceptible-Infected-Recovered) Model

$$\text{Susceptible: } \frac{dS(t)}{dt} = -\beta S(t)I(t)$$

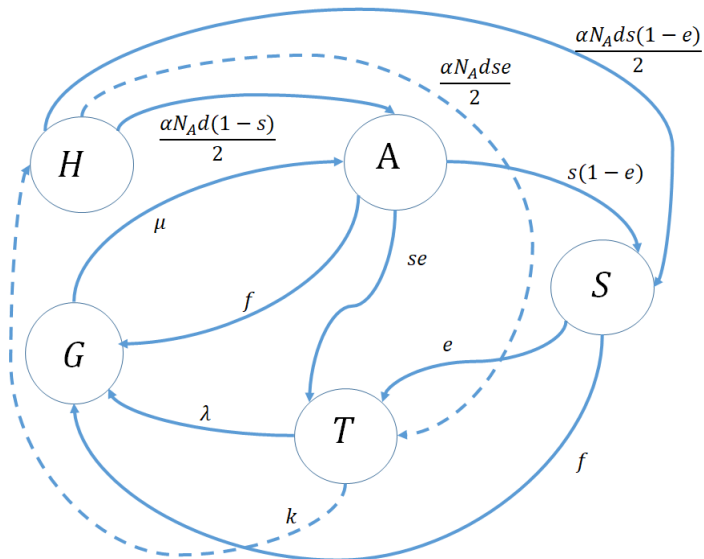
$$\text{Infected : } \frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t)$$

$$\text{Recovered: } \frac{dR(t)}{dt} = \gamma I(t)$$

- Time is continuous, and denoted by t .
- β is the infection rate, and γ is the recovery rate.
- Note that population is stationary.

$$\frac{dS(t)}{dt} + \frac{dI(t)}{dt} + \frac{dR(t)}{dt} = 0$$

Illustration



An Economy Model in Discrete Time

Healthy:

$$N_H(t+1) = N_H(t) - \frac{\alpha N_H(t) N_A(t)}{2} d$$

Infected and don't know:

$$N_A(t+1) = \frac{\alpha N_H(t) N_A(t)}{2} d(1-s) + \mu N_G(t) - (s+f) N_A(t) + N_A(t)$$

Diagnosed positive and light:

$$N_S(t+1) = \frac{\alpha N_H(t) N_A(t)}{2} ds(1-e) + s(1-e) N_A(t) - (e+f) N_S(t) + N_S(t)$$

Diagnosed positive and medically treated:

$$N_T(t+1) = se N_A(t) + e N_S(t) - (k+\lambda) N_T(t) + N_T(t)$$

Recovered and Immune:

$$N_G(t+1) = \lambda N_T(t) + f[N_S(t) + N_A(t)] - \mu N_G(t) + N_G(t)$$

Basic Reproduction Number 1

In the original SIR model,

$$\frac{dI(t)}{dt} = [\beta S(t) - \gamma]I(t)$$

At time 0, the virus spreads if and only if $\beta S(0) - \gamma > 0$, or equivalently,

$$R_0 = \frac{\beta S(0)}{\gamma} > 1$$

where R_0 is called the "Basic Reproduction Number".

Let p be the "Herd Immunity Threshold", which is defined by

$R_0(1 - p) = 1$, or equivalently,

$$p = 1 - \frac{1}{R_0} = \begin{cases} .33, & \text{for } R_0 = 1.5 \\ .50, & \text{for } R_0 = 2 \\ .66, & \text{for } R_0 = 3 \end{cases}$$

Basic Reproduction Number 2

In my model, let $A(t) = N_A(t) + N_S(t) + N_T(t)$. $A(t)$ denotes the mass of infected people. We can write

$$A(t+1) - A(t) = N_A(t) \left[\frac{d\alpha(t)N_H(t) - 2f}{2} \right] + \mu N_G(t) - fN_S(t) - (k + \lambda)A(t)$$

Specifically, at time 0, $N_H(0) = 1$, $N_G(0) = N_S(0) = N_T(0) = 0$. Then

$$A(1) - A(0) = N_A(0) \left[\frac{d\alpha(0) - 2f}{2} \right]$$

Therefore, the basic reproduction number is given by

$$R_0 = \frac{d\alpha(0)}{2f}$$

With $d = .2$, $f = .7$, we obtain

$$\begin{aligned} R_0 &= 2.286, \text{ for } \alpha(0) = 16 \\ &= 1.429, \text{ for } \alpha(0) = 10 \end{aligned}$$

Notations

Basic Idea: Two persons meet and produce economic output, and transfer virus.

Labor Force: $L = N_H + N_A + N_G$, Infected: $S = N_A + N_S + N_T$

and

- α : meeting rate, d : infection rate
- s : rate of being diagnosed positive ($1 - s$: negative)
- f : recovery rate
- e : rate of being heavily ill ($1 - e$: lightly ill)
- k : death rate
- μ : rate of losing antibody (immunodeficient)

The Output

In the service economy, output is produced by contacts by people.

	N_H	N_A	N_G
N_H	y (counted twice)	y	y
N_A	y	y (counted twice)	y
N_G	y	y	y (counted twice)

Let N denote # of people.

There are combinations as many as

$${}_N C_2 = \frac{N(N-1)}{2} \simeq \frac{N^2}{2}, \text{ as } N \text{ is large enough}$$

The appropriate measure for aggregate output is

$$y_t = \frac{\alpha_t (N_H^2 + N_A^2 + N_G^2 + 2N_H N_A + 2N_A N_G + 2N_H N_G)}{2}$$

Baseline Parameters

The baseline parameters are given as follows.

$$\begin{aligned}\rho &= 0, \rho_1 = .1, \rho_2 = .2, \\ \eta &= .1, s = .2, \sigma = 1, \lambda = .4, \mu = 0 \\ d &= .2, e = .3, f = .7, k = .05 \\ \alpha_0 &= 16, \alpha_G = 10\end{aligned}$$

where α_0 is the initial level of meeting rate, while α_G is the target level of meeting rate that is set by the government.

The Meeting Rate

On April 7, 2020 when the government of Japan announced a national emergency, it said that the meeting rate was needed to decrease by 80 percent, while it was reduced only by 30 to 50 percent on average according to the report on the "self-restraint" ratio by NHK. Table shows the week-day self-restraint ratios of large urban area of Japan. The self-restraint ratio is higher in urban area than in rural area. More than 80 percent of the infected population live in the top 10 populated urban area, including the five prefectures listed in the table. Therefore the decrease in the meeting rate from $\alpha_0 = 16$ to $\alpha_G = 10$ is a reasonable parametrization.

Prefecture	5/25	5/18	5/11	4/27	4/20
Hokkaido	20.8	22.8	24.5	27.0	21.9
Tokyo	45.6	49.0	49.1	53.1	53.1
Aichi	20.4	26.1	32.7	31.1	30.9
Osaka	26.2	33.8	35.6	38.2	38.2
Fukuoka	18.8	28.8	30.7	33.3	32.9

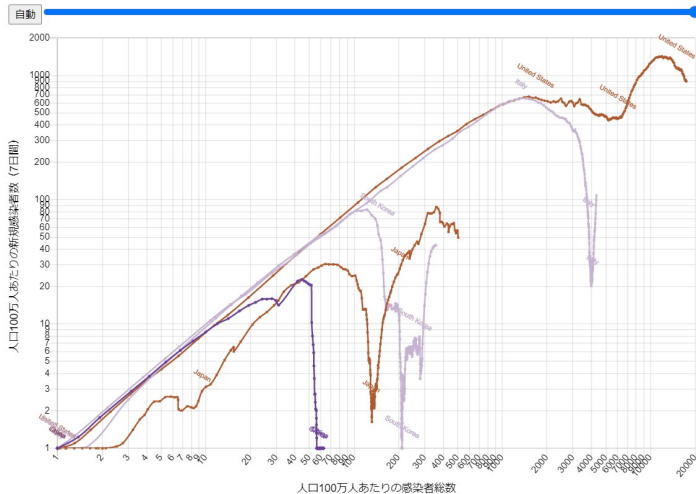
Table: The Self-Restraint Ratio (NHK)

Simulation Scenarios

- Changing death rate (k) does not affect the results so much.
- Any positive losing antibody rate (μ) might destroy human beings.
- The COVID-19 might be even more threatening than HIV, since the former's infection rate is much higher than the latter.
- Simulation has too many scenarios, partly because government takes many prevention measures in the real world.
- Without prevention measures, the final deaths are projected to be as much as 400,000 (comparable to the prediction by the expert community).

Facts as of August 26, 2020

2020/08/26



Horizontal axis: #stock infected per 1 million population

Vertical axis: #flow infected per 1 million population

International Comparison

August 26,2020	Japan	South Korea	Italy	United States
flow	49	42	108	896
stock	504	350	4319	17459
growth rate (%)	9.7	12.0	2.5	5.1

There seem to exist two worlds: The East and the West.

People infected are the higher in the west than in the east, while the growth rate has been much reduced in the west.

In South Korea, the mass of infected people is the lowest.

However, those countries (Japan and South Korea) having successfully put down the initial burst seem to suffer the second attack.

In Europe and the US, the flow and stock of infected people are extremely high, but seem to start to decline.

Baseline Setting

Consider exercises that government reduces the meeting rate from $\alpha_0 = 16$ to $\alpha_G = 10$ with different speeds.

$$\alpha_{t+1} = \alpha_t(1 - \rho) + \rho\alpha_G$$

where $\rho = .1$ or $\rho = .2$ represents the speed of adjustment. The higher ρ is, the faster the reduction is.

- Nt stands for the serious patients to be treated at hospitals.
- c stands for the medical capacity ($c = 0.001$ equals 100 thousands for population 100 millions).
- Beds as much as 100 thousands are not enough to accept patients at the pandemic peak. Beds of only less than 20 thousands are available in Japan.
- Numerical exercises suggest that "The earlier reduction in the contact rate (α) is more effective than the lower target (α_G)."
- The government should take whatever containment policies as early as possible

Lockdown-recovery-lockdown

Consider a lockdown-recovery-lockdown policy such that

$$\alpha_{t+1} = (1 - \rho)\alpha_t + \rho\alpha_G, \text{ for } t < 30$$

$$\alpha_{t+1} = (1 - \eta)\alpha_t + \eta\alpha_0, \text{ for } 30 \leq t < 60$$

$$\alpha_{t+1} = (1 - \rho)\alpha_t + \rho\alpha_G, \text{ for } t \geq 60$$

where

$$\alpha_0 = 16: \text{ initial meeting rate}$$

$$\alpha_G = 10: \text{ meeting rate target}$$

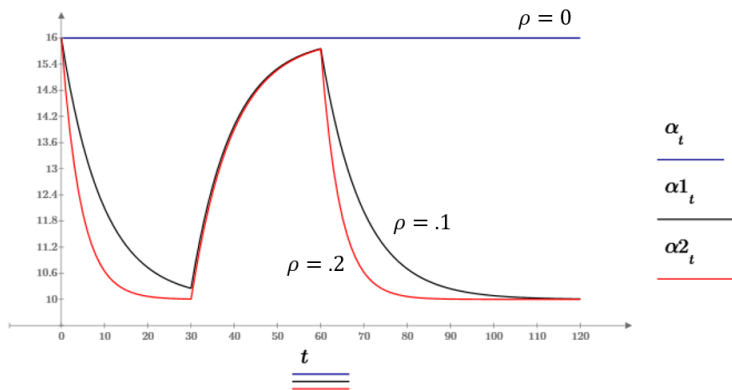
and

$$\rho : \text{ reduction rate}$$

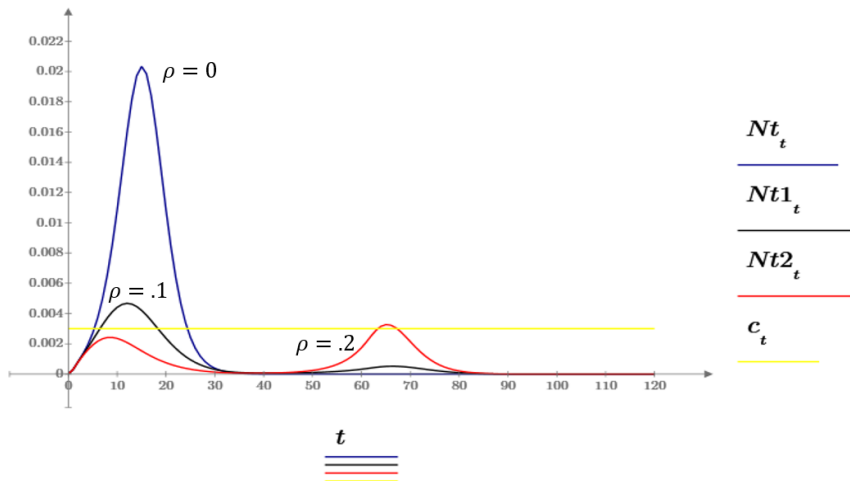
$$\eta = .1: \text{ recovery rate}$$

The Second Lockdown

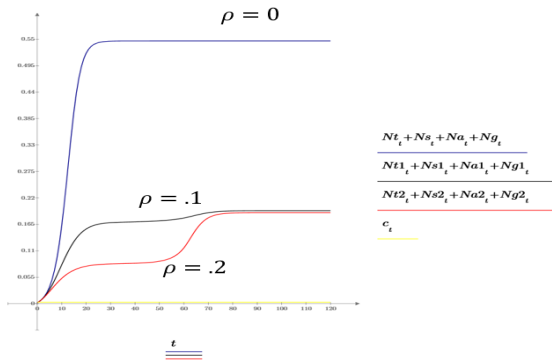
The meeting rate gradually rises from 10 to 16 at time 30, then declines from 16 to 10 at time 60.



Medical Capacity

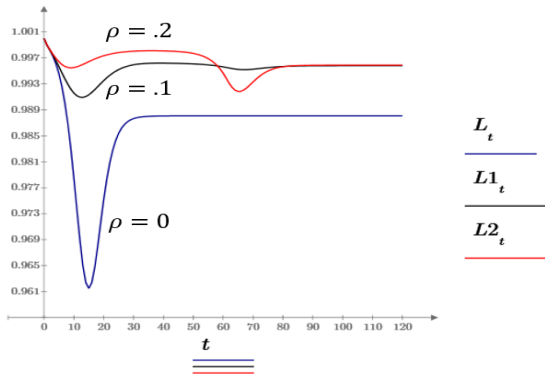


Medical capacity is overflowed with weaker lockdown. Stricter lockdown ($\rho = .2$) preserve medical capacity both in the first and the second

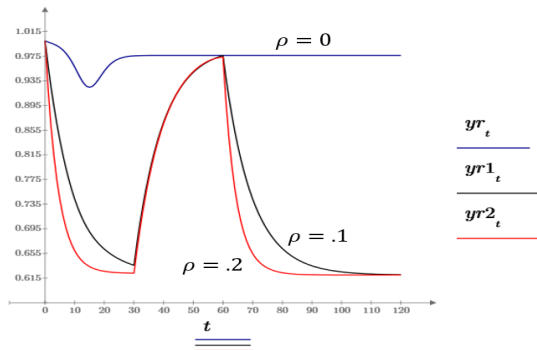


Repeated lockdowns with stricter regulation reduce the required level of herd immunity.

Labor Force



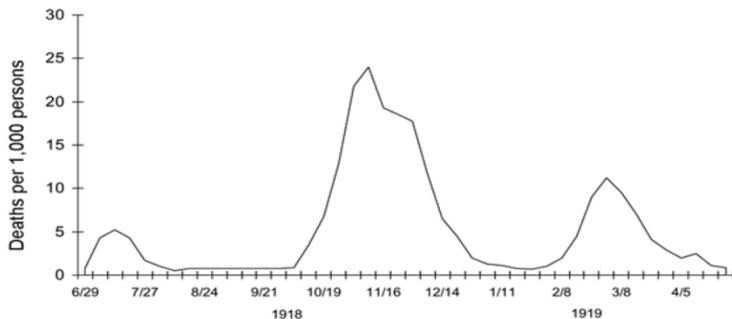
Loss in the labor force is the smallest for the most strict lockdown.



Output loss is the smallest for the case with no lockdown.

The Spanish Flu 1918-1919

Figure 1. Three Pandemic Waves of the Spanish Flu in the United Kingdom, 1918-20



E. Jordan, Epidemic *Influenza a Survey*, Chicago, Ill. American Medical Association, 1927

Figure: <https://www.bauer.uh.edu/centers/irf/houston-updates-march20.php>

Lockdown-recovery-lockdown-recovery

A lockdown-recovery-lockdown-recovery process like the Spanish Flu is simulated. The meeting rate changes as follows.

$$\alpha_{t+1} = (1 - \rho)\alpha_t + \rho\alpha_G, \text{ for } t < 30$$

$$\alpha_{t+1} = (1 - \eta)\alpha_t + \eta\alpha_0, \text{ for } 30 \leq t < 60$$

$$\alpha_{t+1} = (1 - \rho)\alpha_t + \rho\alpha_G, \text{ for } t \geq 60$$

$$\alpha_{t+1} = (1 - \eta)\alpha_t + \eta\alpha_0, \text{ for } 60 \leq t < 90$$

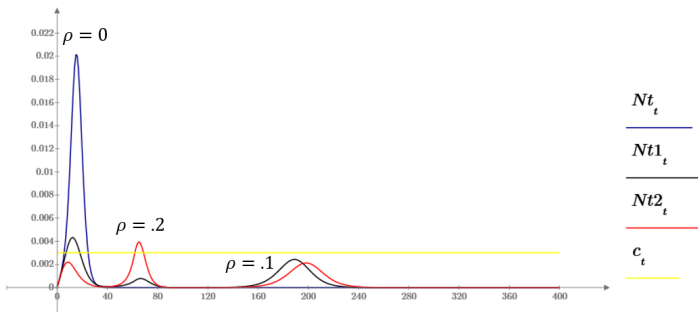
$$\alpha_{t+1} = (1 - \rho)\alpha_t + \rho\alpha_G, \text{ for } t \geq 120$$

where $\rho = 0, .1, .2$ is the parameter representing the speed to the lockdown target, while $\eta = .1$ is the speed of recovery to the initial meeting rate.

$\rho = 0$ is the case that there is no contact control, while the target meeting rate is reached faster for $\rho = .2$ than for $\rho = .1$

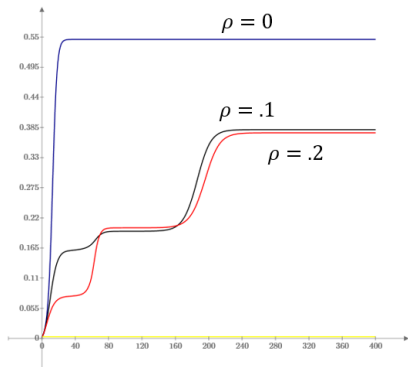
Medical Capacity

There are three peaks. The first peak is the highest for $\rho = 0$ with no lockdown, while it is followed by even higher second and third peak of infection with some lockdown, of which the pattern is similar to the Spanish Flu 1918-1919. The third peak could be higher than the second one, if regulation is weak ($\rho = 1$), while the second peak is the highest for $\rho = 2$ as in the Spanish Flu pandemic.



Immunity

The long-run level of antibody possession is 55% without regulation, and is 38% with regulation. If contact control is as strong as $\rho = .2$, the level of antibody possession is only 6% after the first attack, and the society is vulnerable to the second attack. Herd immunity is achieved at the third attack for $\rho = .1$ and $\rho = .2$.



$$\frac{Nt_t + Ns_t + Na_t + Ng_t}{Nt1_t + Ns1_t + Na1_t + Ng1_t}$$

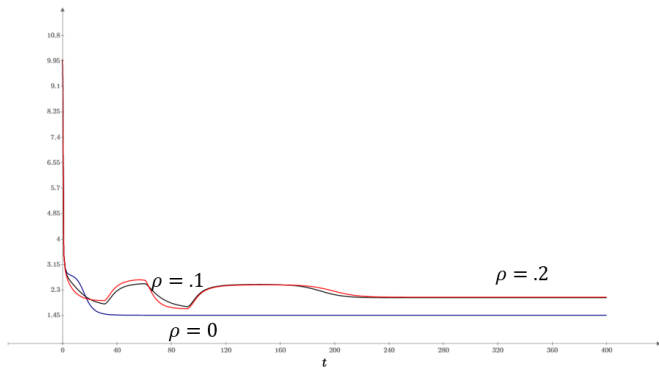
$$\frac{Nt2_t + Ns2_t + Na2_t + Ng2_t}{c_t}$$

Missing the Infectious

The ratio of the infectious agents all to recognized is written as

$$\frac{N_A(t) + N_S(t) + N_T(t)}{N_S(t) + N_T(t)}$$

which soon decreases to less than three even under contact control/avoidance.



$$\frac{N_{a_t} + N_{t_t} + N_{s_t}}{N_{t_t} + N_{s_t}}$$

$$\frac{N_{a1_t} + N_{t1_t} + N_{s1_t}}{N_{t1_t} + N_{s1_t}}$$

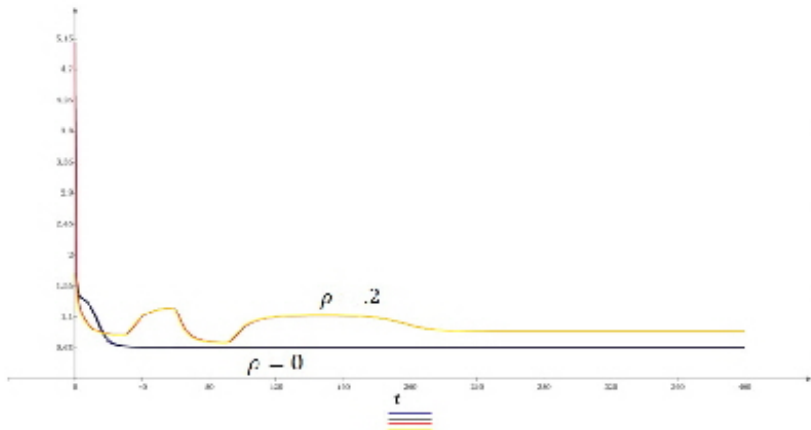
$$\frac{N_{a2_t} + N_{t2_t} + N_{s2_t}}{N_{t2_t} + N_{s2_t}}$$

The gross growth rate of

(1) recognized infectious agents : $\frac{N_S(t+1) + N_T(t+1)}{N_S(t) + N_T(t)}$

(2) unrecognized and unrecognized infectious agents : $\frac{N_A(t+1) + N_S(t+1)}{N_A(t) + N_S(t)}$

There is no significant difference between (1) and (2).



$$\frac{Nt_{t+1} + Na_{t+1}}{Nt_t + Na_t}$$

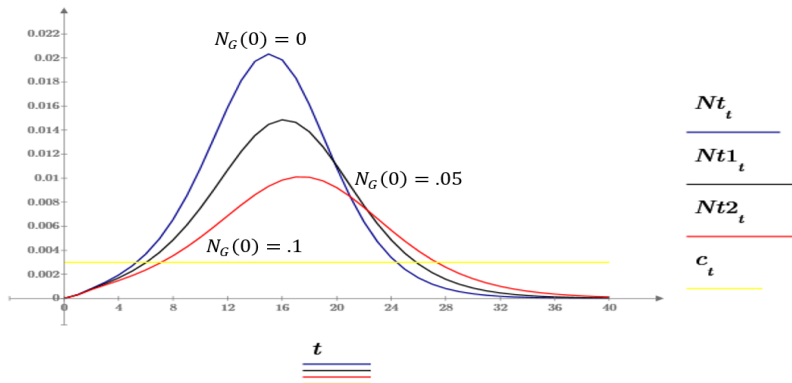
$$\frac{Na_{t+1} + Nt_{t+1}}{Na_t + Nt_t}$$

$$\frac{Nt2_{t+1} + Na2_{t+1}}{Nt2_t + Na2_t}$$

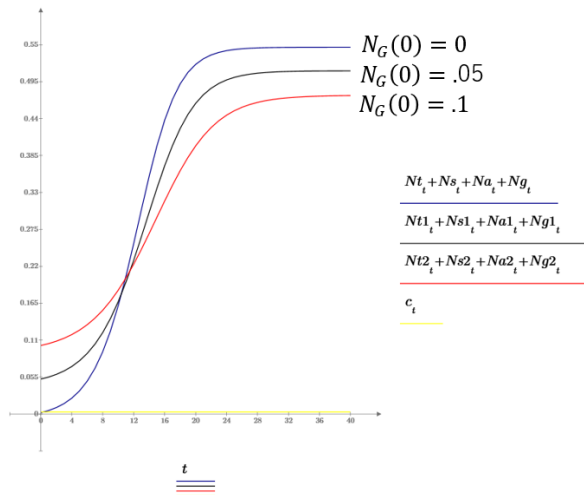
$$\frac{Na2_{t+1} + Nt2_{t+1}}{Na2_t + Nt2_t}$$

Initial Immunity

The three cases with $N_G(0) = 0$, $N_G(0) = .05$, $N_G(0) = .1$ are tested. The rise in initial immunity reduces the speed of infection as well as the burden on medical institutions.

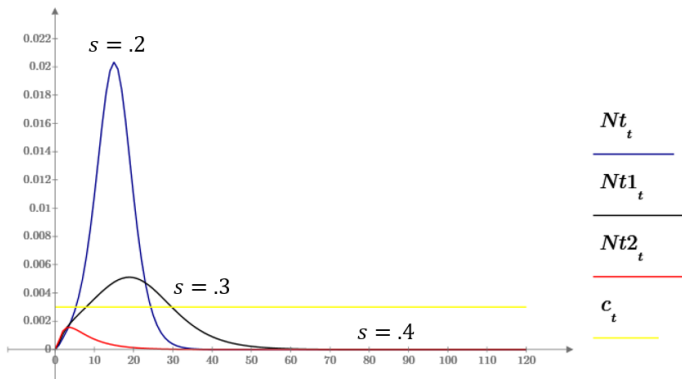


A higher initial immunity reduces the required level of antibody possession at herd immunity.

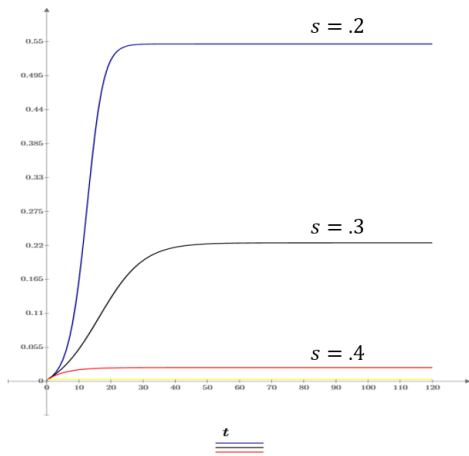


Testing

A higher rate of finding infectious population reduces the chance to be infected. Consider a rise in the probability to find agents infected from $s = .2$ to $s = .4$. The result is striking. Pandemic is shortened by the stricter testing policy without the second strike, while the fraction of people possessing the antibody is only 2% for $s = .4$.

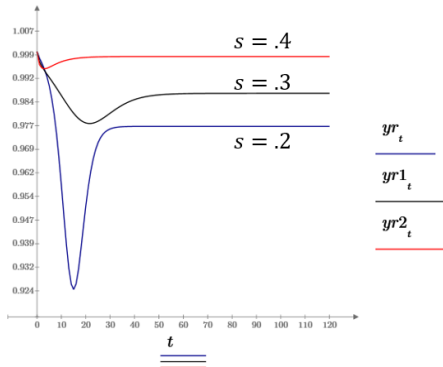
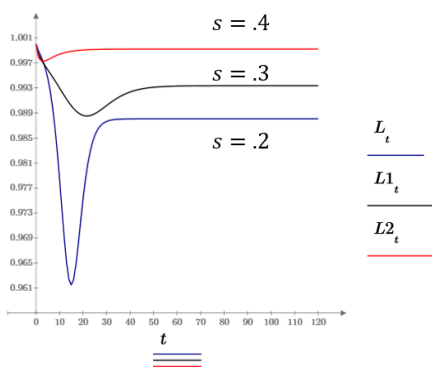


The higher rate of testing shortens the initial attack, but leaves the economy more vulnerable to the virus infection from outside.



$$\frac{\frac{Nt_t + Ns_t + Na_t + Ng_t}{Nt1_t + Ns1_t + Na1_t + Ng1_t}}{\frac{Nt2_t + Ns2_t + Na2_t + Ng2_t}{e_t}}$$

The stricter testing results in the smaller loss in labor force and output. This the potential reason why economic loss is small for South Korea.



Alternative Contact Control

Consider an alternative functional form of the meeting probability. Assume

$$\alpha_t = \alpha_M + (\alpha_0 - \alpha_M) \left(\frac{\pi t}{T} \right)$$

where

α_0 : initial level

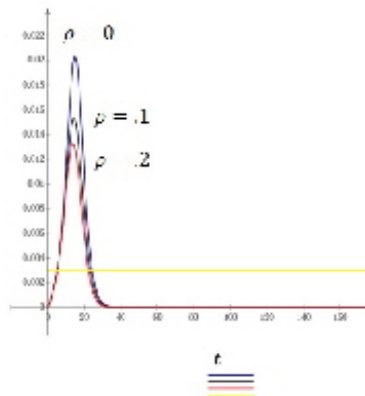
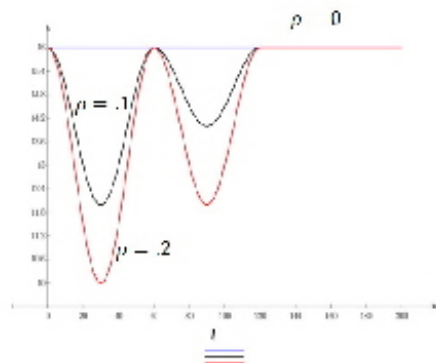
α_M : intermediate target

T : #periods of increase or decrease

For example, consider a time dependent contact-control rule like

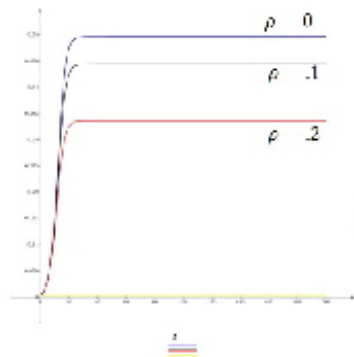
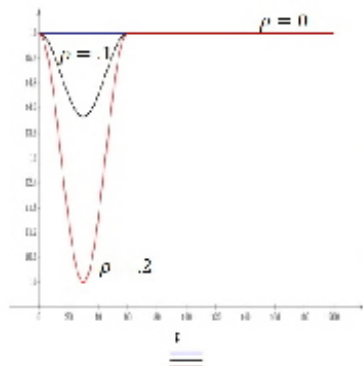
$$\alpha_{1t} = \begin{cases} 14 + 2 \cos\left(\frac{\pi t}{30}\right), & \text{for } 0 \leq t < 60 \\ 15 + \cos\left(\frac{\pi t}{30}\right), & \text{for } 61 \leq t < 120 \\ 16 & \end{cases}, \alpha_{2t} = \begin{cases} 12 + 4 \cos\left(\frac{\pi t}{30}\right), & \text{for } 0 \leq t < 60 \\ 14 + \cos\left(\frac{\pi t}{30}\right), & \text{for } 61 \leq t < 120 \\ 16 & \end{cases}$$

The Cosine Curve



The reduction in the meeting rate dampens as time passes. This meeting probability process produces the population hospitalized that violates the medical capacity even under 50% reduction in the meeting probability.

No need to have a second lockdown

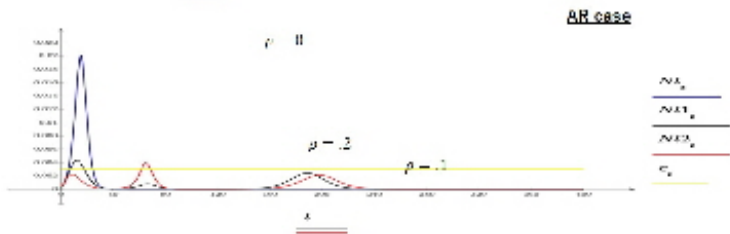
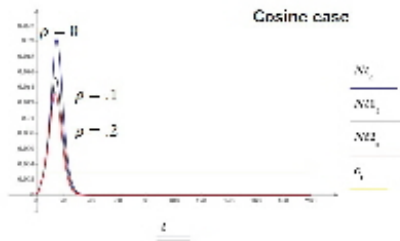


$$\frac{dS_t}{dt} = -\beta S_t I_t + \gamma I_t$$

$$\frac{dI_t}{dt} = \beta S_t I_t - \gamma I_t$$

$$\frac{dD_t}{dt} = \gamma I_t$$

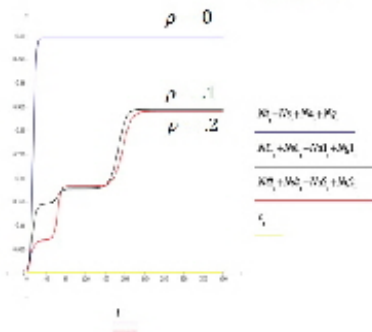
Comparison of the autoregressive case and the cosine curve case: Medical Capacity



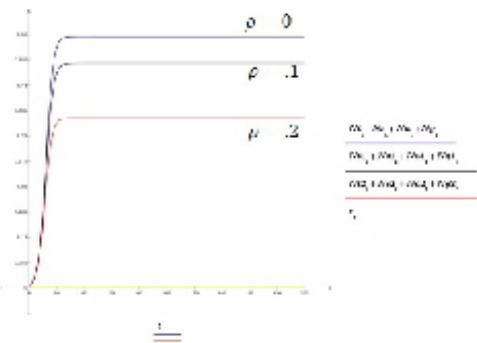
Comparison: Herd Immunity

The implication on herd immunity is as follows. In the long run, the required level of the antibody possession is the same for $\rho = 0$ and $\rho = .2$ between the AR case and the cosine case. However, the required level of the antibody possession for the cosine case is higher than for the AR case for $\rho = 1$.

AR case



Cosine case



Medical Capacity

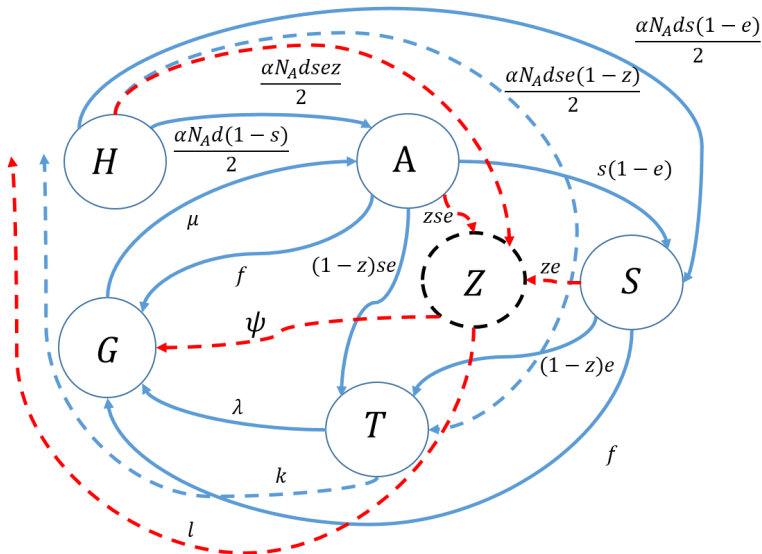
Some newly infected people are rejected by the hospital, due to the capacity limit c .

Self-cared patients recover to be healthy at a lower rate $\psi < \lambda$ and die at a higher rate $l > k$ than patients cared at the hospital.

A new state-dependent parameter z is added to the original model. N_z is the mass of people infected by the virus and rejected by the hospital due to the capacity constraint. z changes as follows.

$$z_{t+1} = \begin{cases} 0, & \text{if } N_t < c \\ 1, & \text{otherwise.} \end{cases}$$

The Model with Medical Capacity: Illustration



The Model with Medical Capacity

$$N_H(t+1) = N_H(t) - \frac{\alpha N_H(t) N_A(t)}{2} d$$

$$N_A(t+1) = \frac{\alpha N_H(t) N_A(t)}{2} d(1-s) + \mu N_G(t) - (s+f) N_A(t) + N_A(t)$$

$$N_S(t+1) = \frac{\alpha N_H(t) N_A(t)}{2} ds(1-e) + s(1-e) N_A(t) - (e+f) N_S(t) + N_S(t)$$

$$N_T(t+1) = \frac{\alpha N_H(t) N_A(t)}{2} dse(1-z_t) + (1-z_t) se N_A(t) + (1-z_t) e N_S(t) + N_T(t)$$

$$N_Z(t+1) = \frac{\alpha N_H(t) N_A(t)}{2} dsez_t + z_t se N_A(t) + z_t e N_S(t) - (\psi + l) N_Z(t) + N_Z(t)$$

$$N_G(t+1) = \lambda N_T(t) + f[N_S(t) + N_A(t)] + \psi N_Z(t) - \mu N_G(t) + N_G(t)$$

The Susceptible Agents

Let $V_H, V_A, V_S, V_T, V_Z, V_G$ be the value functions of the states. They satisfy the Bellman equations as follows.

$$V_H(t) = \left(1 - \frac{\alpha N_A(t)}{2}\right) \beta V_H(t+1): \text{meet whom I can work with}$$
$$+ \frac{\alpha N_A(t) + \alpha N_H(t) + \alpha N_G(t)}{2} u(y): \text{meet whom I can work with}$$
$$+ \frac{\alpha N_A(t)}{2} \beta \left[\begin{array}{l} (1-d)V_H(t+1): \text{not infected} \\ +d(1-s)V_A(t+1): \text{infected but not recognized} \\ +ds(1-e)V_S(t+1): \text{infected and not so serious} \\ +dse(1-z)V_T(t+1): \text{infected and cared at a hospital} \\ +dsezV_Z(t+1): \text{infected and cared at home} \end{array} \right]$$

The Infectious Agents

$$V_A(t) = \frac{\alpha N_A(t) + \alpha N_H(t) + \alpha N_G(t)}{2} u(y): \quad \text{meet whom I}$$
$$+\beta[1 - s(1 - e) - zse - (1 - z)se - f]V_A(t + 1): \text{ no change occurs}$$
$$+\beta \left[\begin{array}{l} s(1 - e)V_S(t + 1): \text{ showing light symptoms} \\ +(1 - z)seV_T(t + 1): \text{ showing serious symptoms and going to a hosp} \\ +zseV_Z(t + 1): \text{ showing serious symptoms and cared at a hospital} \\ +fV_G(t + 1): \text{ recovered to be immune} \end{array} \right]$$

The Other Values

$$V_S(t) = [1 - (1 - z)e - ze - f]\beta V_S(t + 1) \\ + \beta[(1 - z)eV_T(t + 1) + zeV_Z(t + 1) + fV_G(t + 1)]$$

$$V_T(t) = \beta[1 - \lambda - k]V_T(t + 1) + \beta[\lambda V_G(t + 1) + kV(D)]$$

$$V_Z(t) = (1 - \psi - l)V_Z(t + 1) + \beta[\psi V_G(t + 1) + lV(D)]$$

$$V_G(t) = \frac{\alpha N_A(t) + \alpha N_H(t) + \alpha N_G(t)}{2} u(y) + \beta[(1 - \mu)V_G(t + 1) + \mu V_A]$$

People act forward-looking. Then the value functions are solved by backward induction. Solving value functions is equivalent to solving a terminal value problem, while the distribution of people is solved as a usual initial-value problem.

The Steady State Value

The economy reaches a steady state at time T at which

$$N_A(t) = 0, \quad N_H(t) + N_G(t) < 1, \quad \text{for } t \geq T$$

with the steady state value functions satisfying

$$V_H(T) = \frac{\alpha N_H(T) + \alpha N_G(T)}{2} u(y) + \beta V_H(T)$$

$$V_A(t) = \frac{\alpha N_H(T) + \alpha N_G(T)}{2} u(y) \\ + \beta [1 - s(1 - e) - zse - (1 - z)se - f] V_A(T) \\ + \beta [s(1 - e)V_S(T) + (1 - z)seV_T(T) + zseV_Z(T) + fV_G(T)]$$

$$V_G(T) = \frac{\alpha N_H(T) + \alpha N_G(T)}{2} u(y) + \beta [(1 - \mu)V_G(T) + \mu V_A(T)]$$

$$V_S = \frac{\beta [(1 - z)eV_T + zeV_Z + fV_G]}{(1 - z)e + ze + f}, \quad V_T = \frac{\beta [\lambda V_G + kV(D)]}{\lambda + k}, \quad V_Z = \frac{\beta [\psi V_G]}{\psi + \lambda}$$

The Meeting Rate

The meeting rate follows a repeated hammer-and-dance process.

$$\alpha_{t+1} := \left\| \begin{array}{l} \text{if } t < 30 \\ \quad \left\| \alpha_t \cdot (1 - \rho) + \rho \cdot \alpha g \right. \\ \text{if } t > 29 \\ \quad \left\| (1 - \eta) \cdot \alpha_t + \eta \cdot \alpha_1 \right. \\ \text{if } 59 < t < 90 \\ \quad \left\| \alpha_t \cdot (1 - \rho) + \rho \cdot \alpha g \right. \\ \text{if } t > 90 \\ \quad \left\| (1 - \eta) \cdot \alpha_t + \eta \cdot \alpha_1 \right. \end{array} \right\|$$

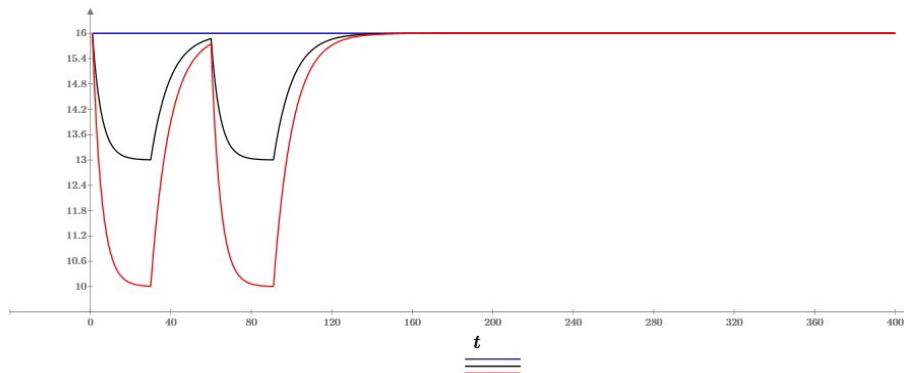
Terminal Condition

The terminal values at time T are solved as

$$\begin{array}{rcl} Vh_T & & 158.72 \\ Va_T & & 154.15 \\ Vs_T & & 141.54 \\ Vt_T & = & 126.31 \\ Vz_T & & 90.02 \\ Vg_T & & 158.72 \end{array}$$

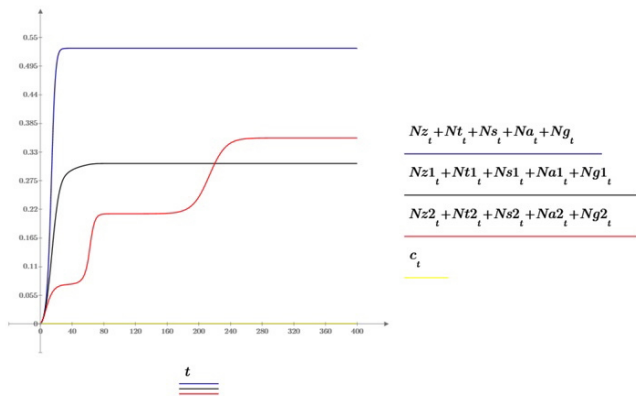
where Vz_T denotes the value of being ill without being hospitalized.
The file: corona047-value-dynamics-correct.

Shallow or Deep?



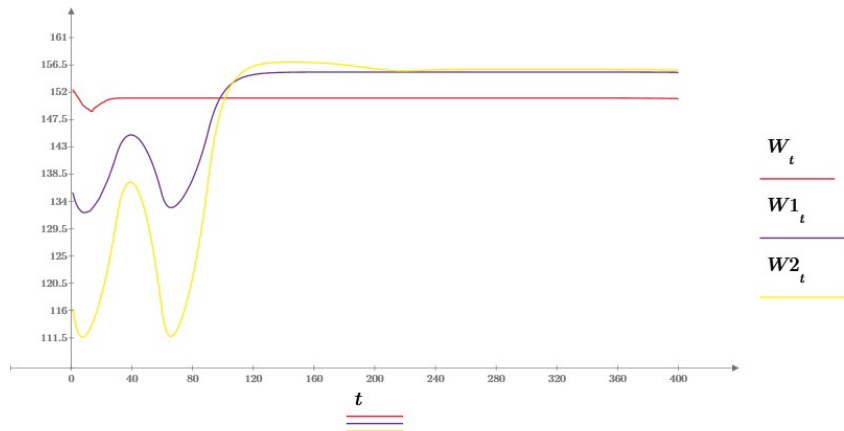
The meeting rate is maintained at $\alpha = 16$, or goes down to $\alpha_1 = 13$ (loose regulation), or $\alpha_2 = 10$ (tight regulation). There is a medical capacity $c = .001$.

Immunity



Herd Immunity is achieved at lower level of infection if people are protected by a tighter regulation. This is a major benefit of regulation on mobility.

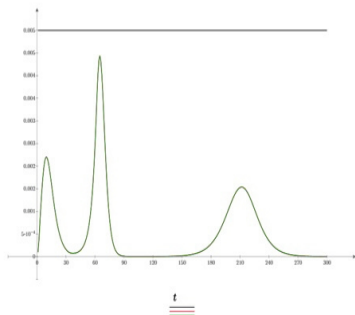
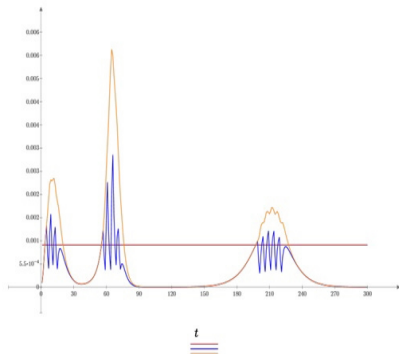
Welfare Consequence of the Lockdown



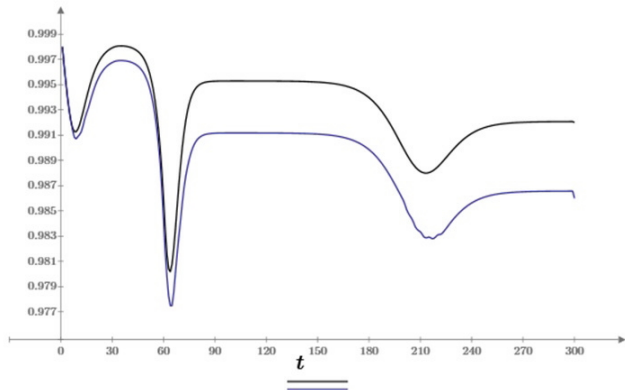
The welfare gap is not so large between $\alpha_1 = 13$ and $\alpha_2 = 10$, suggesting that a tighter regulation is not rewarded by enough gains.

Medical Capacity

Consider medical capacity constraint $c_0 = .001$ or $c_1 = .005$, under which pandemic occurs three times. Under $c_0 = .001$, some infected people are not hospitalized.

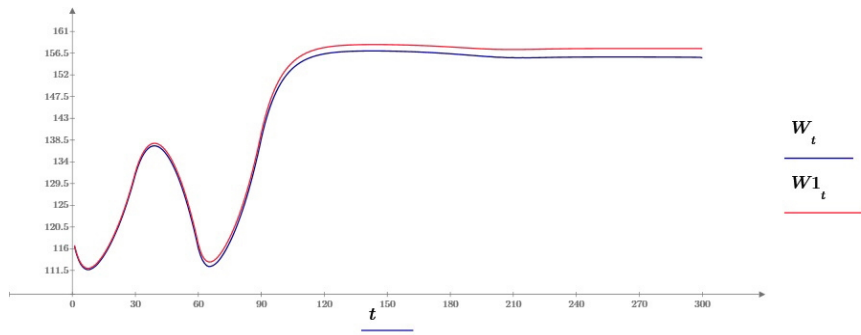


Larger medical capacity reduces labor force loss



$$\frac{Nh_t + Ng_t}{Nh_t + Ng_t}$$

Medical capacity makes no big difference in welfare



Summary 1

- 1 There is a trade-off: Stronger lockdown policy restricts the mass of people infected within medical capacity in the first strike, but immunity remains at a relatively low level, and the community is vulnerable to the upcoming virus strike.
- 2 A higher level of initial immunity (like BCG) reduces the required immunity level necessary to stop pandemic.
- 3 Repeated containment policy might be effective to reduce intertemporal loss of lives and outputs by raising the initial level of immunity at each virus strike.
- 4 More intensive testing shortens the duration of the first strike, but makes the community more vulnerable to the second strike.

Summary 2

- ① After all, no or less mobility control will result in more long-term workforce loss, but in smaller short-term economic loss.
- ② In order to evaluate economic welfare, the question is how to take the lost utility of dead people into account.
- ③ The total utility goes down as the population declines.
- ④ Can we evaluate economic welfare appropriately, by simply adding the lifetime utility of people alive?

Reference



Hiroshi Nishiura and Hisashi Inaba (2006): Prediction of Infectious Disease Outbreak with Particular Emphasis on the Statistical Issues Using Transmission Model, *Proceedings of the Institute of Statistical Mathematics* Vol. 54, No. 2, 461–480 (in Japanese)



Wikipedia: Compartmental models in epidemiology,
https://en.wikipedia.org/wiki/Compartmental_models_in_epidemiology