

# A Marriage Model with Directed Search

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# Introduction: Marriage Models

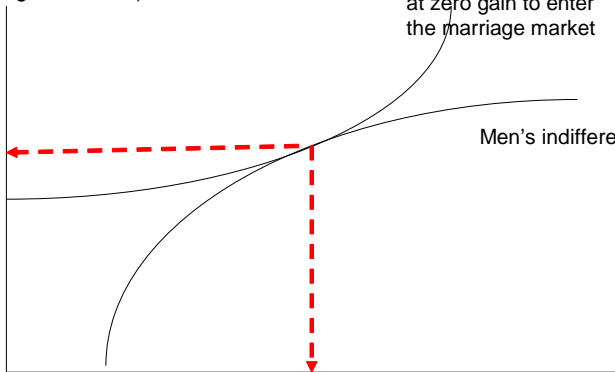
- Becker - frictionless match
  - PAM (positive assortative matching): Good (Bad) men are matched with Good (Bad) women.
- Search Models 1 - frictional match with random matching
  - Burdett & Coles - emergence of social classes
  - Shimer & Smith clarify the conditions to yield PAM.
  - complex, difficult to solve
- Search Models 2 - frictional match with directed search
  - Men (Women) attract women (men) by posting marriage conditions
  - A version of hedonic price models (price-quality competition)

# Homogeneous Agents

Wife's share of marriage output  
("marriage condition")

Women's indifference curve  
at zero gain to enter  
the marriage market

Men's indifference curve



Degree of competition  
= ratio of women to men

# The Model with homogenous men and women

Single men and women searching for partners

$$\begin{aligned}(r + \delta)U_M &= -k_M + \theta\alpha(\theta)[J_M - U_M] \\ (r + \delta)U_W &= -k_W + \alpha(\theta)(J_W - U_W)\end{aligned}$$

Matched men and women

$$\begin{aligned}(r + \delta)J_M &= x - w + \delta(U_M - J_M) + \sigma(U_M - J_M - c) \\ (r + \delta)J_W &= w - z + \delta(U_W - J_W) + \sigma(U_W - J_W + c)\end{aligned}$$

There are men and women, each of which has of population unity. All men are in the marriage market.  $\theta$  is the ratio of women to men in the marriage market, representing the 'tightness' of the marriage market.

# Solution

In the original scenario, women post *marriage condition(s)* ( $w$ ). Men apply for women. Women enter the marriage market until the value of searching equals the value of being out of the marriage market.

Women solve

$$\max_w U_W \quad \text{subject to } U_M \geq \bar{U}_M$$

which is equivalent to

$$\max_w U_M \quad \text{subject to } U_W \geq 0$$

# Heterogeneous Men and Women

- A man earns  $x$  and a woman earns  $y$ .
- A fraction  $l$  of the women's income is lost in marriage.
- There is another extra cost  $z$  for a woman to spend in marriage.
- $m$  denotes the output of marriage.
- $c$  denotes the severance transfer, which is paid from a husband to his wife when they divorce.
- All the income and marriage output are transferable.
- While in marriage, a man obtains  $x + y(1 - l) + m - w$  and a woman obtains  $w - z$ .
- The fraction of highly productive men is  $n_M$ , while the fraction of highly productive women is  $n_W$ . I assume that  $n_M > n_W$ .

$\alpha(\theta)$  = probability for a woman to match a man

$\theta\alpha(\theta)$  = probability for a man to match a woman

$$\theta = \frac{\text{number of women}}{\text{number of men}}$$

$$\alpha'(\theta) < 0, \eta(\theta) \equiv \frac{\theta\alpha'(\theta)}{\alpha(\theta)} \in (-1, 0)$$

## Alternative Models

- Model 1: Women post marriage conditions  $w$  and men apply for women.
- Model 2: Men post marriage conditions  $w$  and women apply for men.



# Model 1 - Sequence of Actions

- 1 Men perceive a market tightness  $\theta$  for each set  $(x, y)$ , and chooses the market to enter.
- 2 Taking  $\theta$  as given, each woman decides if she searches for a partner or not.
- 3 If a woman enters a marriage market, she announces the wife's payoff  $w$ .
- 4 Men apply after observing all the women's choices.
- 5 Each woman selects one of the men who apply for her. If there is no application, she remains single.

The model is solved by backward induction.

# Value Functions

A man (or woman) of type  $i$  searches for a marriage partner of type  $j$ . The value functions of men and women out of the marriage market are written as follows.

$$(r + \delta)S_{M,i} = x_i, \quad (r + \delta)S_{W,i} = y_i$$

where  $i = H, L$ . The value functions of men and women in searching for marriage partners are written as

$$(r + \delta)U_{Mi} = x_i - k_M + \theta\alpha(\theta)[J_{Mij} - U_{Mi}]$$

$$(r + \delta)U_{Wi} = x_i - k_W + \alpha(\theta)[J_{Wij} - U_{Wi}]$$

The value functions of men and women in marriage are written as follows.

$$(r + \delta)J_{Mij} = x_i + y_j(1 - l) + m - w_{ij} + \delta(U_{Mi} - J_{Mij}) + \sigma(U_{Mi} - J_{Mij} -$$

$$(r + \delta)J_{Wij} = w - z + \delta(U_{Wi} - J_{Wij}) + \sigma(U_{Wi} - J_{Wij} + c)$$

The gain to search for men and women are written as

$$\Sigma_{Mij} = -k_M + \theta\alpha(\theta)[J_{Mij} - U_{Mi}]$$

$$\Sigma_{Wij} = -k_W + \alpha(\theta)[J_{Wij} - U_{Wi}]$$

# Solution

A woman  $j$  in the marriage market  $(i, j)$  solves

$$\max_{w_{ij}} U_{Wj}, \quad \text{subject to } U_{Mi} \geq S_{Mi}$$

The constraint is equivalent to

$$\Sigma_{Mij} \equiv -k_M + \theta\alpha(\theta)[J_{M,ij} - U_{M,i}] \geq 0$$

In an equilibrium with directed search, the problem is identical to

$$\max_{w_{ij}} U_{Mj}, \quad \text{subject to } \Sigma_{Wij} \geq 0$$

In other words, competition among women induces them to post the lowest payoff at which they were indifferent between in and out of the marriage market. Men choose which market to enter by comparing  $\Sigma_{Mij}$  among different types of women.

# The model is tractable!

An equilibrium is characterized by the two conditions.

- 1 the tangency condition of the indifference curve of the men on search to the the iso-gain curve for women to enter the marriage market

$$w_{ij} = \left[ \frac{1 + (1 - l)\Psi}{1 + \Psi} \right] y_j + \frac{\Psi(m + k_M)}{1 + \Psi} + \frac{z - k_W}{1 + \Psi} - \sigma c$$

where

$$\Psi = - \left( 1 - \frac{1}{\eta(\theta)} \right) \frac{r + 2\delta + \sigma + \alpha(\theta)}{r + 2\delta + \sigma + \theta\alpha(\theta)} > 0$$

- 2 the gain for women to enter the marriage market equals zero.

$$w_{ij} = y_j + \left[ \frac{r + 2\delta + \sigma}{\alpha(\theta)} \right] k_W + z - c$$

## Model 2 - Sequence of Actions

- 1 Women perceive a market tightness  $\theta$  for each set  $(x, y)$ , and chooses the market to enter.
- 2 Taking  $\theta$  as given, each man decides if he searches for a partner or not.
- 3 If a man enters a marriage market, he announces the wife's payoff  $w$ .
- 4 Women apply after observing all the men's choices.
- 5 Each man selects one of the women who apply for her. If there is no application, he remains single.

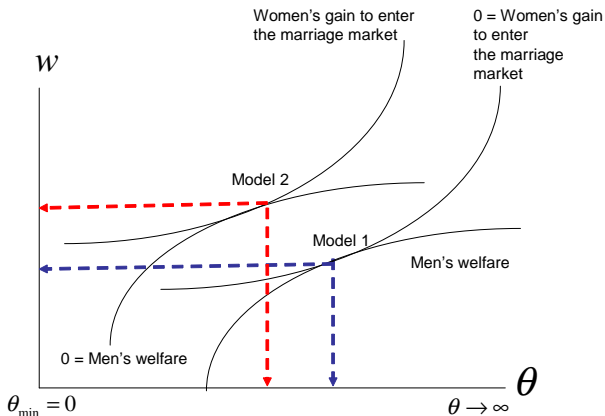
The model is solved by backward induction.

# Equilibrium with a inner solution

In the model 1 equilibrium, the gain for women to enter the marriage market equals zero through competition among women, while men can choose the market that yields the higher gain to them. Then a fraction of women do not participate in the marriage market, while all the men participate in it.

In the model 2 equilibrium, the gain for men to enter the marriage market equals zero through competition among men, while women can choose the market that yields the higher gain to them. Then a fraction of men do not participate in the marriage market, while all the women participate in it.

# Equilibrium with a inner solution - graph



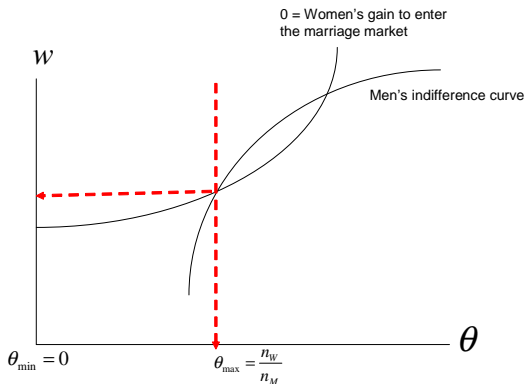
Recall the stable matching theory. Marriage institution that men proposes to women creates more advantage to men than women.

# Equilibrium with a corner solution

- If marriage is so productive that the gain for all the women (or men) to posts the marriage condition and participate in the marriage market is nonnegative, there is a constrained equilibrium with a corner solution.
- In the model 1 equilibrium, marriage market tightness  $\theta$  can take a value from zero to  $n_W / n_M$  for a given  $n_M$ . At the corner solution,  $\theta = \frac{n_W}{n_M}$ , and the marriage condition  $w$  is determined by the women's zero gain condition to participate.
- In the model 2 equilibrium, marriage market tightness  $\theta$  can take a value from  $n_W / n_M$  to infinity for a given  $n_W$ . At the corner solution,  $\theta = \frac{n_W}{n_M}$ , and the marriage condition  $w$  is determined by the men's zero gain condition to participate.

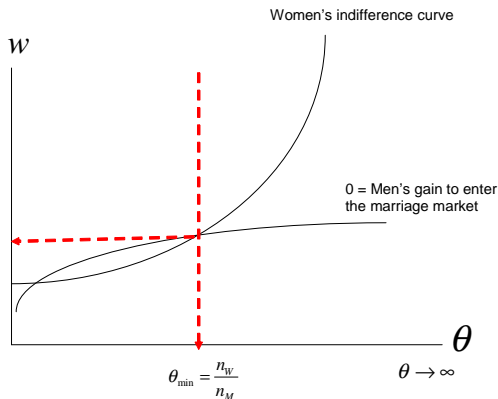


# Model 1 - a corner solution



**Figure:** Corner Solution 1: The women's zero gain condition is binding, and all the men and women participates in the marriage market.

## Model 2 - a corner solution



**Figure:** Corner Solution 1: The men's zero gain condition is binding, and all the men and women participates in the marriage market.

I consider three potential equilibria.

- ① PAM1: High (low) quality men are matched with high (low) quality women. Some men or women stay out of the marriage market.
- ② PAM2: High quality men are matched with high quality women. All the men and women participate in the marriage market.
- ③ NAM1: High (low) quality men are matched with low (high) quality women. Some men or women stay out of the marriage market.
- ④ NAM2: High quality men are matched with low quality women. All the men and women participate in the marriage market.

# Positive Assortative Matching - the men proposing game

- With inner solution, the equilibrium set  $(\theta, w)$  is determined by the tangency condition and the zero gain condition.
- In the PAM, a high-quality man will marry a high-quality woman at rate  $\theta_{HH}\alpha(\theta_{HH})$  and pays wage  $w_{HH}$ . Agents might obtain an extra gain by deviating from the equilibrium strategy. A high quality men can deviate: He could marry a low-quality woman at rate  $\theta_{LL}\alpha(\theta_{LL})$  and pays wage  $w_{LL}$ . The low-quality woman would accept a high-quality man. A low-quality man, however, might not be able to deviate. A high-quality woman might reject a man if she found his quality low.
- In the present model, however, both types of men can deviate, since women obtain zero gains to search partners and receive the equilibrium wage.

Upperscript  $D$  denotes deviation. A PAM exists if and only if

$$U_{MHH} > U_{MHL}^D, \text{ and } U_{MLL} > U_{MLH}^D$$

which is equivalent to

$$\Sigma_{MHH} > \Sigma_{MHL}^D, \text{ and } \Sigma_{MLL} > \Sigma_{MLH}^D$$

where

$$\Sigma_{Mij} = \frac{\theta\alpha(\theta)[(1-l)y_j - w_{ij} + (m - \sigma c)] - (r + 2\delta + \sigma)k_M}{r + 2\delta + \sigma + \theta\alpha(\theta)}$$

and subscripts to  $\theta$  and  $m$  are omitted.

The women's zero gain condition yields the equilibrium wage as

$$w_{ij} = y_j + \frac{r + 2\delta + \sigma}{\alpha(\theta)} k_W - \sigma c + z$$

Using this expression we can eliminate  $w_{ij}$  from  $\Sigma_{Mij}$  to obtain the equilibrium gain to search as

$$\Sigma_{Mij} = \frac{\theta\alpha(\theta)[-ly_j + (m - z)] - (r + 2\delta + \sigma)(k_M + \theta k_W)}{r + 2\delta + \sigma + \theta\alpha(\theta)}$$

Therefore, the gain for men to participate in the marriage market is

- 1 a weighted average of the surplus of marriage  $-ly_j + (m - z)$ , and the search cost  $-(k_M + \theta k_W)$ , and
- 2 independent of the type of the partner, if  $l = 0$  and  $m - z$  does not depend on the type of agents.

- If  $l > 0$  and  $m - z$  is common to all the married couples, the gain for men to participate in the marriage market is decreasing in the partner's productivity. Matching with a more productive woman will incur a larger income loss upon marriage.
- Therefore the income loss  $-ly_j$  must be more than compensated by a higher marriage rate  $\theta\alpha(\theta)$ , if the equilibrium matching is positive-assortative.

# The General Case where marriage output is increasing in the types of men and women

- Following the Becker's tradition, we assume that

$$m_{HH} > m_{HL} = m_{LH} > m_{LL}$$

That is, marriage output shows complementarity of the types of partners.

- In deviation, applicants are randomly chosen to be husbands/wives, no matter what their types are. A high-type man is as likely to be chosen to be a husband as a low-type man. This is a natural consequence of the setting that
  - 1 the wife's payoff  $w$  depends only on the market condition, not on the specific type of her husband, and
  - 2 the marriage condition is not renegotiated.



**Proposition 1A:** The matching is positive-assortative, if and only if

$$\Sigma_{MHH} > \Sigma_{MHL}^D \text{ and } \Sigma_{MLL} > \Sigma_{MLH}^D$$

where upperscript  $D$  denotes the deviation, and

$$\Sigma_{MHH} = \frac{\theta_{HH}\alpha(\theta_{HH})[-ly_H + (m_{HH} - z)] - (r + 2\delta + \sigma)(k_M + \theta_{HH}k_W)}{r + 2\delta + \sigma + \theta_{HH}\alpha(\theta_{HH})}$$

$$\Sigma_{MHL}^D = \frac{\theta_{LL}\alpha(\theta_{LL})[-ly_L + (m_{HL} - z)] - (r + 2\delta + \sigma)(k_M + \theta_{LL}k_W)}{r + 2\delta + \sigma + \theta_{LL}\alpha(\theta_{LL})}$$

$$\Sigma_{MLL} = \frac{\theta_{LL}\alpha(\theta_{LL})[-ly_L + (m_{LL} - z)] - (r + 2\delta + \sigma)(k_M + \theta_{LL}k_W)}{r + 2\delta + \sigma + \theta_{LL}\alpha(\theta_{LL})}$$

$$\Sigma_{MLH}^D = \frac{\theta_{HH}\alpha(\theta_{HH})[-ly_H + (m_{LH} - z)] - (r + 2\delta + \sigma)(k_M + \theta_{HH}k_W)}{r + 2\delta + \sigma + \theta_{HH}\alpha(\theta_{HH})}$$

## Proof.

Consider a deviation by a high-quality man from his equilibrium strategy. His gain to stay in the high-quality market is

$$\Sigma_{MHH} = \frac{\theta_{HH}\alpha(\theta_{HH})[-ly_H + (m_{HH} - z)] - (r + 2\delta + \sigma)(k_M + \theta_{HH}k_W)}{r + 2\delta + \sigma + \theta_{HH}\alpha(\theta_{HH})}$$

His gain to deviate and search in the low-quality market is

$$\Sigma_{MHL}^D = \frac{\theta_{LL}\alpha(\theta_{LL})[(1-l)y_L - w_{LL} + (m_{HL} - \sigma c)] - (r + 2\delta + \sigma)k_M}{r + 2\delta + \sigma + \theta_{LL}\alpha(\theta_{LL})}$$

The wage the man considers is the wage that prevails in the low-quality market, and his partner's income is of the low type. We can eliminate  $w_{LL}$  by using the zero gain condition of the low-type women

$$w_{LL} = y_L + \frac{r + 2\delta + \sigma}{\alpha(\theta)} k_W - \sigma c + z$$

to obtain

- There are three factors to determine the strategies of men.
  - ①  $m - z$  : the marriage output.
  - ②  $-ly$  : the loss of the women's income, caused by marriage.
  - ③  $\theta$  : the tightness of the marriage market.
- With a corner solution, the equilibrium shows the similar characteristics. The wage is determined by the women's zero gain condition, and the men take all the marriage surplus net of the income loss, while the market tightness is exogenously given.

# Negative Assortative Matching - the men proposing game

**Proposition 1B:** The matching is negative-assortative, if and only if

$$\Sigma_{MHL} > \Sigma_{MHH}^D \text{ and } \Sigma_{MLH} > \Sigma_{MLL}^D$$

where

$$\begin{aligned}\Sigma_{MHL} &= \frac{\theta_{HL}\alpha(\theta_{HL})[-ly_L + (m_{HL} - z)] - (r + 2\delta + \sigma)(k_M + \theta_{HL}k_W)}{r + 2\delta + \sigma + \theta_{HL}\alpha(\theta_{HL})} \\ \Sigma_{MHH}^D &= \frac{\theta_{LH}\alpha(\theta_{LH})[-ly_H + (m_{HH} - z)] - (r + 2\delta + \sigma)(k_M + \theta_{LH}k_W)}{r + 2\delta + \sigma + \theta_{LH}\alpha(\theta_{LH})} \\ \Sigma_{MLH} &= \frac{\theta_{LH}\alpha(\theta_{LH})[-ly_H + (m_{LH} - z)] - (r + 2\delta + \sigma)(k_M + \theta_{LH}k_W)}{r + 2\delta + \sigma + \theta_{LH}\alpha(\theta_{LH})} \\ \Sigma_{MLL}^D &= \frac{\theta_{HL}\alpha(\theta_{HL})[-ly_L + (m_{LL} - z)] - (r + 2\delta + \sigma)(k_M + \theta_{HL}k_W)}{r + 2\delta + \sigma + \theta_{HL}\alpha(\theta_{HL})}.\end{aligned}$$

# Positive Assortative Matching - the women proposing game

The gain for a woman of type  $j$  to search for a man type  $i$  is obtained as

$$\Sigma_{Wij} = \frac{\alpha(\theta_{ij})[-y_j + (m_{ij} - z)] - (r + 2\delta + \sigma)(k_M/\theta_{ij} + k_W)}{r + 2\delta + \sigma + \alpha(\theta_{ij})}$$

Suppose now that a positive assortative matching prevails. Then the equilibrium gain for a high type woman to search for a high type man is

$$\Sigma_{WHH} = \frac{\alpha(\theta_{HH})[-ly_H + (m_{HH} - z)] - (r + 2\delta + \sigma)(k_M/\theta_{HH} + k_W)}{r + 2\delta + \sigma + \alpha(\theta_{HH})}.$$

Consider a deviation strategy for a high type woman to search for a low type man. She enters the marriage market with tightness  $\theta_{LL}$ , where she receives output  $w_{LL}$ . The gain to deviate is computed as

$$\Sigma_{WLH}^D = \frac{\alpha(\theta_{LL})[-y_H + \sigma c - z + w_{LL}] - (r + 2\delta + \sigma)k_W}{r + 2\delta + \sigma + \alpha(\theta_{LL})}.$$

The wage is determined in the low-type market and given by

$$w_{LL} = (1 - l)y_L + m_{LL} - \sigma c - \frac{r + 2\delta + \sigma}{\theta_{LL}\alpha(\theta_{LL})}k_m$$

Note that  $m_{LL}$  shows up in  $w_{LL}$ , but not in  $\Sigma_{WLH}^D$ . By eliminating  $w_{LL}$  in  $\Sigma_{WLH}^D$ , we obtain

$$\Sigma_{WLH}^D = \frac{\alpha(\theta_{LL})[(-y_H + y_L) - ly_L + (m_{LL} - z)] - (r + 2\delta + \sigma)(k_M + \theta_{LL}k_W)}{r + 2\delta + \sigma + \alpha(\theta_{LL})}$$

She does not deviate if and only if  $\Sigma_{WHH} > \Sigma_{WLH}^D$ .

**Proposition 2A:** The equilibrium matching is positive assortative if and only if

$$\Sigma_{WHH} > \Sigma_{WLH}^D \text{ and } \Sigma_{WLL} > \Sigma_{WHL}^D$$

where  $\Sigma_{WLL}$  and  $\Sigma_{WHL}^D$  are similarly defined.

Similarly, we obtain

**Proposition 2B:** The equilibrium matching is negative assortative if and only if

$$\Sigma_{WLH} > \Sigma_{WHH}^D \text{ and } \Sigma_{WHL} > \Sigma_{WLL}^D$$

**Proposition 3:** For women, given other conditions equal, the equilibrium gain is larger than the deviation gain by  $y_H - y_L$ .

## Remark: Externality

- Compare  $\Sigma_{WHH}$  and  $\Sigma_{WLH}^D$ , neglecting the common search costs, to obtain

$$\Sigma_{WHH} = \frac{\alpha(\theta_{HH})[-ly_H + (m_{HH} - z)]}{r + 2\delta + \sigma + \alpha(\theta_{HH})}$$
$$\Sigma_{WLH}^D = \frac{\alpha(\theta_{LL})[(-y_H + y_L) - ly_L + (m_{LL} - z)]}{r + 2\delta + \sigma + \alpha(\theta_{LL})}.$$

- $y_H - y_L$  is a positive externality that the high-type wife produces for the low-type husband.
- There is no externality for a high-type man to produce in deviation. The extra productivity due to the improvement of the husband type is internalized and exploited by the man.
- On the contrary, if a high-type woman switches from the high to low market, the extra gain due to the improvement of the wife type is taken by the husband.



## Remark: Institution matters

- The externality is not realized in the equilibrium. However, it reduces the high-type women's incentive to deviate, and increases the low-type women's incentive to deviate in the positive-assortative matching.
- This is an artifact from the directed search setting that the “wage” (marriage condition) is determined by the market, and the extra gain or loss is not redistributed to women.
- In an alternative world, husbands might receive fixed marriage income, and wives might take the residual. Such a marriage institution tends to affect the men's incentive to deviate from the current matching regime.

- I have built a tractable model of marriage with directed search.
- The results depend on the two institutional settings.
  - ① Who proposes and who accepts/rejects.
  - ② Who manage household budget.
- Next: we can find the parameter areas to support each type of equilibrium.