

The Ramsey Model

Ryoichi Imai

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1 Preparation

1.1 Discounting in continuous time

Remember that the exponential function is defined as

$$e^x = \left[\lim_{h \rightarrow 0} (1 + h)^{\frac{x}{h}} \right].$$

In order to know the present discounted value (PDV) of the utility at time t , we divide t into t/Δ subperiods, each of which has Δ of time length. Then the PDV of the utility at time t is given by

$$u_t(1 - r\Delta)^{\frac{t}{\Delta}} = u_t(t + h)^{-\frac{rt}{h}}$$

where $h = -r\Delta$. Now we consider infinitesimally short periods, that is, $\Delta \rightarrow 0$. Then $h \rightarrow 0$, and we have

$$\lim_{\Delta \rightarrow 0} u_t(1 - r\Delta)^{\frac{t}{\Delta}} = u_t \lim_{h \rightarrow 0} (t + h)^{-\frac{rt}{h}} = u_t e^{-rt} \equiv u_t \exp(-rt).$$

Similarly, the PDV of utility at time t discounted from present time s is obtained as

$$u_t e^{-r(t-s)} \equiv u_t \exp[-r(t-s)].$$

1.2 Euler's Formula

A function $f(x_1, x_2, \dots, x_n)$ is *homogeneous of degree* r if

$$f(tx_1, tx_2, \dots, tx_n) = t^r f(x_1, x_2, \dots, x_n), \quad \text{for any } r > 0.$$

If a function is homogeneous of degree r and differentiable, we have

$$\sum_{n=1}^N \frac{\partial f(x_1, x_2, \dots, x_N)}{\partial x_n} = r f(x_1, x_2, \dots, x_N),$$

which is called *Euler's Formula*.

Proof of the Euler's Formula : By definition,

$$f(tx_1, tx_2, \dots, tx_N) = t^r f(x_1, x_2, \dots, x_N), \quad \text{for any } r > 0.$$

Differentiating this expression with respect to t gives

$$\sum_{n=1}^N \frac{\partial f(x_1, x_2, \dots, x_N)}{\partial x_n} x_n = r t^{r-1} f(x_1, x_2, \dots, x_N)$$

Evaluating at $t = 1$, we obtain Euler's Formula. ■

The most important application of Euler's formula is the CRS (homogeneous of degree one) production function, $F(K, L)$. Euler's Formula says

$$\frac{\partial F(K, L)}{\partial K} K + \frac{\partial F(K, L)}{\partial L} L = F(K, L)$$

In terms of per capita capital, $k = K/L$, we have

$$r = f'(k), \quad w = f(k) - k f'(k).$$

In other words, in the Walrasian equilibrium, if the production function is homogeneous of degree one (or equivalently, shows constant returns to scale), the output is totally distributed to production factors, and there is no extra surplus.

2 The intertemporal resource constraint

The economy produces a single output, which can be consumed and invested, with the aggregate production function

$$Y_t = F(K_t, B_t N_t),$$

where Y_t is output, K_t is capital, N_t is labor, and B_t is the level of technology. Define

$$c = \frac{C_t}{B_t N_t}, \quad k_t = \frac{K_t}{B_t N_t}$$

Dividing Y_t by $B_t N_t$, we have the per capita production function as

$$y_t \equiv \frac{Y_t}{B_t N_t} = \frac{F(K_t, B_t N_t)}{B_t N_t} = F\left(\frac{K_t}{B_t N_t}, 1\right) \equiv f(k_t)$$

The capital accumulation of this economy is described by the transitional equation,

$$\frac{dK_t}{dt} = F(K_t, B_t N_t) - C_t - \delta K_t$$

where δ denotes capital depreciation rate. Define

$$\gamma = \frac{dB_t/dt}{B_t}, \quad n = \frac{dN_t/dt}{N_t}$$

Dividing by $B_t N_t$, we have the transitional equation per capita as follows.

$$\begin{aligned} \frac{dK_t}{dt} \frac{1}{B_t N_t} &= \frac{d(k_t B_t N_t)}{dt} \frac{1}{B_t N_t} \\ \frac{dK_t}{dt} \frac{1}{B_t N_t} &= \frac{F(K_t, B_t N_t)}{B_t N_t} - \frac{C_t}{B_t N_t} - \delta \frac{K_t}{B_t N_t} \\ \frac{B_t N_t dk_t/dt + k_t N_t dB_t/dt + k_t B_t dN_t/dt}{B_t N_t} &= f(k_t) - c_t - \delta k_t \\ \frac{dk_t}{dt} + k_t \frac{dB_t/dt}{B_t} + k_t \frac{dN_t/dt}{N_t} &= f(k_t) - c_t - \delta k_t \end{aligned}$$

Then we obtain the per capita capital accumulation equation as

$$\frac{dk_t}{dt} = f(k_t) - c_t - (\delta + \gamma + n)k_t$$

We can derive the intertemporal resource constraint of this economy as follows. Multiplying the budget constraint by $\exp(-rt)$ and integrate the

resulting expression from $t = 0$ to $t = T$, we have

$$\begin{aligned}
LHS &= \int_0^T \frac{dk_t}{dt} \exp(-rt) dt \\
&= [k_t \exp(-rt)]_0^T + \int_0^T k_t \exp(-rt) dt \\
&= k_T \exp(-rT) - k_0 + r \int_0^T k_t \exp(-rt) dt \\
&= \int_0^T f(k_t) \exp(-rt) dt - \int_0^T c_t \exp(-rt) dt \\
&\quad - (\delta + \gamma + n) \int_0^T k_t \exp(-rt) dt \\
&= RHS
\end{aligned}$$

Then the intertemporal resource constraint with time horizon T is

$$\begin{aligned}
\int_0^T c_t \exp(-rt) dt &= k_0 - k_T \exp(-rT) + \int_0^T f(k_t) \exp(-rt) dt \\
&\quad - (r + \delta + \gamma + n) \int_0^T k_t \exp(-rt) dt.
\end{aligned}$$

As T becomes large to infinity, we obtain the intertemporal budget constraint for the infinite horizon problem as

$$\begin{aligned}
\int_0^\infty c_t \exp(-rt) dt &= k_0 + \int_0^\infty f(k_t) \exp(-rt) dt \\
&\quad - (r + \delta + \gamma + n) \int_0^\infty k_t \exp(-rt) dt. \quad (1)
\end{aligned}$$

where we assume that

$$\lim_{T \rightarrow \infty} k_T \exp(-rT) = 0. \quad (2)$$

Intuitively, it would not be optimal to end up at infinity with positive capital stock because it could be consumed. This condition is rewritten as the *transversality condition* that

$$\lim_{T \rightarrow \infty} u'(c_T) k_T \exp(-rT) = 0 \quad (3)$$

The transversality condition (3) is identical to (2) if the marginal utility from consumption is positive at the end time, $u'(c_T) = 0$. The transversality condition is imposed on the solution of the social planner's maximization problem, which is also called the "Command Optimum".

The intertemporal resource constraint (1) can be read as follows. The left hand side represents the PDV of consumption flows. The right hand side is the sum of the capital stock at time 0 and the PDV of income flows from time 0 to infinity, net of the preferential (r), physical (δ), technological (γ), and demographic (n) depreciation of per capita capital in efficiency units.

3 The Command Optimum

The discounted sum of the utility flows from time s to infinity is written as

$$U_0 = \int_0^{\infty} u(c_t) \exp[-r(t-s)] dt.$$

The social planner maximizes the representative individual's intertemporal utility U_s subject to the budget constraint,

$$\frac{dk_t}{dt} = f(k_t) - c_t - (\delta + \gamma + n)k_t \quad (4)$$

In order to solve this problem, we use the Maximum Principle, in which we distinguish the *control* variables and *state* variables. In this problem, c_t is the control variable, and k_t is the state variable.

First, we write the current value Hamiltonian function as follows.

$$H_t = u(c_t) + \lambda_t [f(k_t) - c_t - (\delta + \gamma + n)k_t]$$

where λ_t is called *co-state* variables. The Maximum Principle consists of two conditions.

1. The first-order conditions for maximization of the Hamiltonian, with respect to the control variables;

$$\frac{\partial H_t}{\partial c_t} = 0.$$

2. The Hamiltonian Dynamics;

$$\frac{d\lambda_t}{dt} = r\lambda_t - \frac{\partial H_t}{\partial k_t}$$

The Hamiltonian Dynamics means that the flow value of each co-state variable, which represents the *shadow price* of the associated state variable, equals the marginal change of the value of the Hamiltonian with respect to the associated state variable.

The Maximum Principle for this problem is obtained as

$$u'(c_t) = \lambda_t \quad (5)$$

$$\frac{d\lambda_t}{dt} = \lambda_t[(r + \delta + \gamma + n) - f'(k_t)] \quad (6)$$

The command optimum is characterized by (c_t, k_t, λ_t) that satisfy the Maximum Principle (5) and (6), and the transitional equation (4) with the transversality condition (3). From (5) and (6), we can eliminate λ_t to obtain

$$\frac{c_t u''(c_t)}{u'(c_t)} \frac{dc_t/dt}{c_t} = (r + \delta + \gamma + n) - f'(k_t) \quad (7)$$

The balanced growth path of this economy (k^*, c^*) , which is the steady state, is obtained by setting $dc_t/dt = dk_t/dt = 0$ as,

$$f'(k^*) = r + \delta + \gamma + n \quad (8)$$

$$c^* = f(k^*) - (\delta + \gamma + n)k^* \quad (9)$$

The dynamics is characterized by (4) and (7).

We can illustrate the steady state and the dynamics of the model in a (k, c) plane of Figure 1, where k is the . The intersection of $dc_t/dt = 0$ and $dk_t/dt = 0$ gives the unique steady state, E . There is a unique equilibrium path, which is called a *saddle path*. All the points on the saddle path satisfy the equilibrium conditions (4), (7), and (3). With rational expectation, once the economy leaves the equilibrium path, consumption jumps to a point on the new equilibrium path.

Let us consider the effects of a sudden decline in TFP growth rate (γ) . A lower γ causes a rightward shift of $dc_t/dt = 0$, while it causes an upward shift of $dk_t/dt = 0$. Therefore the steady state switches from E to F , as is illustrated in Figure 2. A decline in TFP growth leads to a higher c

and k in the new steady state. The equilibrium transaction path is a little complicated. First, the control variable c is instantaneously adjusted and the economy jumps from E to G . Next, the control variable c and the state variable k move slowly to the new steady state F on the new saddle path. Although both c and k rise in the new steady state, consumption and capital per capita change differently. By definition,

$$\frac{C}{N} = Bc, \quad \frac{K}{N} = Bk.$$

On the balanced growth path, the effect of the higher c or k is cancelled out by the lower B . We can illustrate this by looking at the ratio of consumption to GDP;

$$\frac{C}{Y} = \frac{c}{f(k)} = \frac{f(k) - (\delta + \gamma + n)k}{f(k)} = 1 - \frac{(\delta + \gamma + n)k}{f(k)}$$

As k rises in the new steady state, $(\delta + \gamma + n)k/f(k)$ increases due to the homogeneity of degree one of the production function. Therefore a higher fraction of GDP is invested in capital. The ratio of consumption to GDP either rises or declines since a higher $k/f(k)$ is cancelled out by a lower γ . In other words, a lower TFP growth raises capital-output ratio, but its effect on the consumption-output ratio is ambiguous.

A rise in capital-output ratio has been observed in Japan since the crash of the bubble economy in 1990. This is a reason why we can see the slump of Japanese economy as a sign of its productivity growth decline.

4 Decentralization

The command optimum is decentralized by the market mechanism. That is, the allocation in the Walrasian equilibrium is identical to the command optimum. See Blanchard-Fisher for details.

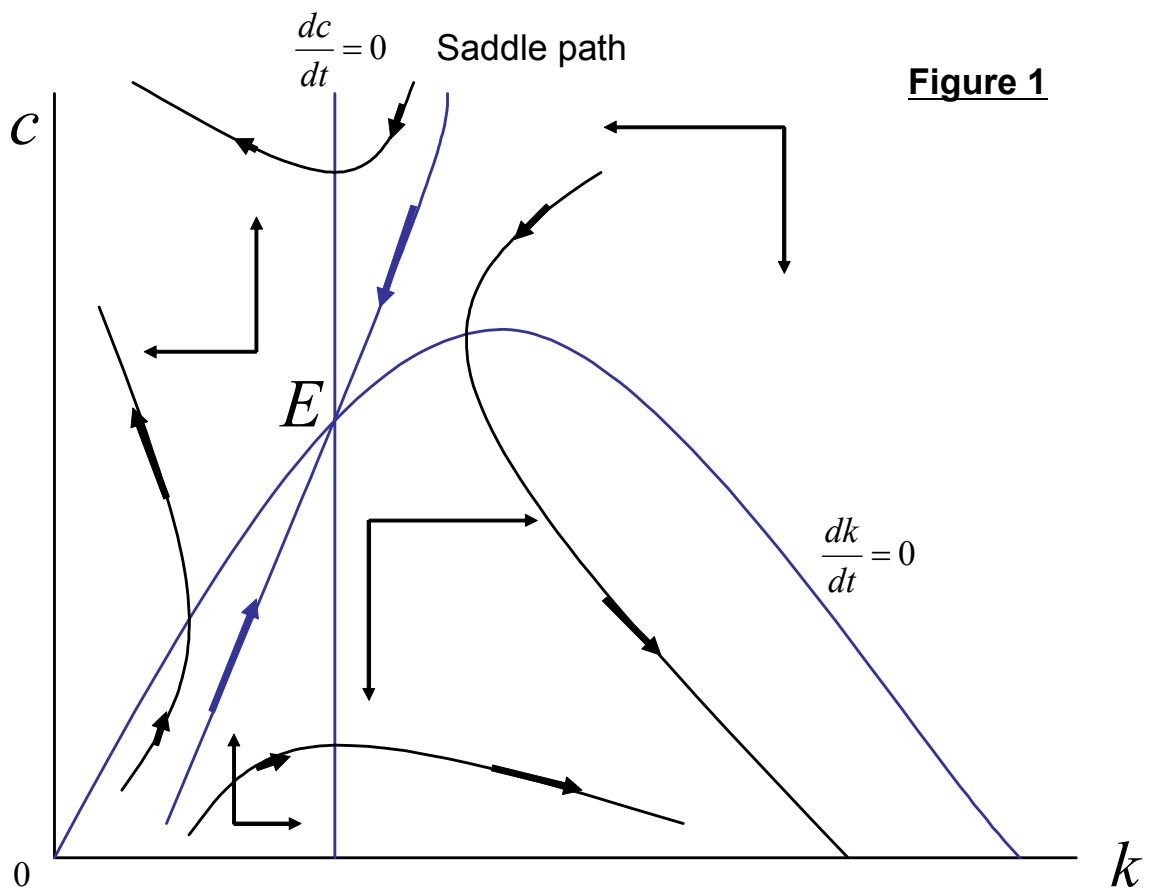


Figure 1

Figure 1:

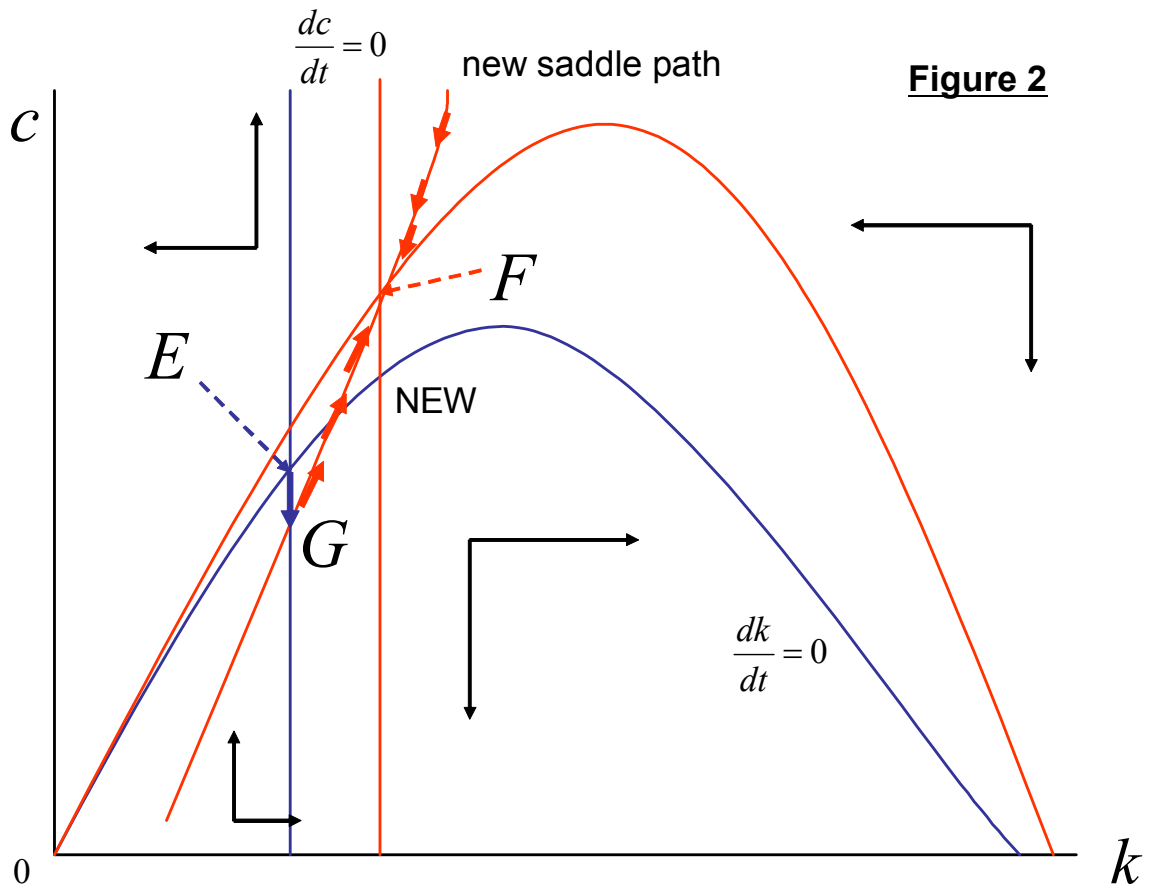


Figure 2

Figure 2: