

Anonymity and equilibrium indeterminacy in a random matching model with money: a conjecture

Nao Sahashi

Osaka Prefecture University, Japan

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Many random matching models with divisible money have exhibited equilibrium indeterminacy.

- Green and Zhou (1998,2002)
- Zhou (1999,2003)
- Kamiya and Shimizu (2006,2011)
- Ishihara (2010)
- Kamiya and Sato (2004)...

Introduction

- The extent and mechanism of the indeterminacy, however, are still not fully understood.
- In this talk, I wanted to establish a new proposition concerning the indeterminacy
- but I have not completed the proof yet. (Perhaps I was wrong...)
- So I present my conjecture and approach.

Conjecture

Consider a random matching model with divisible medium of exchange in which each pair of matched individuals plays an anonymous game. If there exists a stationary monetary equilibrium with certain regularity properties, then there also exists a continuum of such equilibria. Furthermore, these equilibria differ one from another in the aggregate marginal valuation of the medium of exchange.

Model (1)

- An economy populated by a continuum of individuals.
- Time is continuous.
- Infinitely divisible medium of exchange, called money.
- Aggregate distribution of money holdings, $H(m)$, is common knowledge.
- Individuals' money holdings are private information.
- Individuals are matched pairwise according to a Poisson process with parameter θ .
- Matched individuals play anonymous games.

Model (2)

- Each match consists of one individual as a buyer and one as a seller.
- The positions of the buyer and seller are assigned with equal probabilities.
- Buyers have a set of feasible actions X .
- Sellers have a set of feasible actions Y .

Model (3)

In a match with a buyer playing $x \in X$ and a seller playing $y \in Y$,

- the buyer receives $u(x, y)$ units of utility.
- the seller suffers $c(x, y)$ units of disutility.
- the buyer pays $r(x, y)$ units of money to the seller.
- In order to assure the non-negativity of his post-trade money balances, the buyer holding m units of money needs to choose x subject to $R(x) \leq m$.

Assumption (1)

- $X = Y = [0, \infty)$.
- The functions $u(x, y)$, $x \in X$, $y \in Y$, is a non-negative, twice continuously differentiable, bounded function such that $u(0, y) = 0$ for all y and $u_x(x, y) > 0$ and $u_{xx}(x, y) < 0$ for all $x > 0$ and $y > 0$. The function $c(x, y)$, $x \in X$, $y \in Y$, is a non-negative, twice continuously differentiable function such that $c(x, 0) = 0$ for all x and $c_y(x, y) > 0$ and $c_{yy}(x, y) > 0$ for all $x > 0$ and $y > 0$.

Assumption (2)

- The function $r(x, y)$, $x \in X$, $y \in Y$, is a non-negative, twice continuously differentiable function such that $r(0, y) = r(x, 0) = 0$ for all x and y and $r_x(x, y) > 0$, $r_{xx}(x, y) \geq 0$, $r_y(x, y) > 0$ and $r_{yy}(x, y) \leq 0$ for all $x > 0$ and $y > 0$.
- $R(x)$, $x \in X$, is a non-negative twice differentiable function such that $R(0) = 0$, $R'(x) > 0$, $R''(x) \geq 0$ for all $x > 0$ with the property that if $R(x) \leq m$, then, $m - r(x, y) \geq 0$ for all y .

Model (6)

By anonymity we refer to the property that $u(x, y)$, $c(x, y)$ and $r(x, y)$ do not have arguments representing the money holdings of the individuals concerned and the property that the individuals' money holdings and trade histories are private information.

Model (7)

- Individuals discount future utility at a constant discount rate $\rho > 0$ and seek to maximize his expected value of the sum of discounted future net utility.

Fredholm's alternative theorem (part.)

Suppose that the integral equation

$$f(s) = \int K(s, t)f(t)dt \quad (1)$$

with a continuous kernel $K(s, t)$ defined over some intervals of s and t , admits one or more of linearly independent continuous solutions for $f(s)$. Then, the associated equation

$$g(s) = \int K(t, s)g(t)dt \quad (2)$$

also admits the same number of linearly independent continuous solutions for $g(s)$. In addition, in general, $\int f(s)g(s)ds \neq 0$ for each pair of solutions.

Approach

- 1 Construct a mapping each of whose fixed points is a stationary equilibrium.
- 2 Assume that the mapping has a fixed point.
- 3 Linearize the mapping around the fixed point to obtain a mapping from the space of perturbations to itself.
- 4 Find out a solution for the associated equation for this linearized mapping.
- 5 Apply Fredholm's alternative theorem to show the existence of a solution for the linearized mapping.
- 6 If such solutions exist for the mappings linearized at each point in the neighborhood of the fixed point and they satisfy certain regularity conditions, then, an 'integral curve' passing through the fixed point can be drawn.
- 7 The curve must be a continuum of fixed points!

Notation (1)

- $x(m)$: action taken as a buyer by an individual holding m units of money
- x : a function from $M = [0, \infty)$ to $X = [0, \infty)$.
- X : Space of x .
- $y(m)$: action taken as a seller by an individual holding m units of money
- y : a function from $M = [0, \infty)$ to $Y = [0, \infty)$.
- Y : Space of y .

Notation (2)

- $V(m)$: the maximum value of the future utility expected by an individual holding m units of money
- $H(m)$: the proportion of individuals holding m units or less of money
- $v(m) = V'(m)$, $h(m) = H'(m)$.
- v : a function from $M = [0, \infty)$ to $[0, \infty)$.
- h : a function from $M = [0, \infty)$ to $[0, 1]$.
- V : the space of v (not that of V)
- H : the space of h (not that of H)

Stationary equilibrium (1)

Condition for stationarity of money distribution

Define

$$\phi(m, n, o; x, y) = \begin{cases} 0 & \text{if } n - r(x(n), y(o, t)) > m \\ 1 & \text{if } n - r(x(n), y(o)) \leq m \end{cases} \quad (3)$$

and

$$\psi(m, n, o; x, y) = \begin{cases} 1 & \text{if } n + r(x(o), y(n)) > m \\ 0 & \text{if } n + r(x(o), y(n)) \leq m. \end{cases} \quad (4)$$

Money distribution is stationary if

$$0 = \theta \int_{n=m}^{\infty} \int_{o=0}^{\infty} \phi(m, n, o; x, y) h(n) h(o) do dn \\ - \theta \int_{n=0}^m \int_{o=0}^{\infty} \psi(m, n, o; x, y) h(n) h(o) do dn. \quad (5)$$

Condition for aggregate money holdings

$$\int mh(m)dm = \mathcal{M} \quad (6)$$

where \mathcal{M} is the money endowment of the economy, exogenously given.

Stationary equilibrium (3)

Condition for individual optimization

$x(m) \in X$ must maximize

$$\int [u(x, y(n)) + V(m - r(x, y(n)))] h(n) dn.$$

subject to $x \leq R(m)$ and $y(m) \in Y$ maximizes

$$\int [-c(x(n), y) + V(m + r((n), y))] h(n) dn$$

under the constraint $x \leq R(m)$

Stationary equilibrium (4)

Condition for individual optimization

$$\int \left[\begin{array}{l} u_x(x(m), y(n)) \\ -v(m - r(x(m), y(n)))r_x(x(m), y(n)) \end{array} \right] h(n)dn - \lambda(m)R_x(x(m)) = 0, \quad (7)$$

$$\begin{aligned} \lambda(m) &\geq 0, \\ m - R(x(m)) &\geq 0, \\ \lambda(m)(m - R(x(m))) &= 0 \end{aligned} \quad (8)$$

$$\int \left[\begin{array}{l} -c_y(x(n), y(m)) \\ +v(m + r(x(n), y(m)))r_y(x(n), y(m)) \end{array} \right] h(n)dn = 0 \quad (9)$$

Stationary equilibrium (5)

Condition for stationarity of the value function

$$\begin{aligned} &(\rho + \theta)V(m) \\ &= \frac{\theta}{2} \int \left[\begin{array}{l} u(x(m), y(n)) \\ +V(m - r(x(m), y(n))) \\ -c(x(n), y(m)) \\ +V(m + r(x(n), y(m))) \end{array} \right] h(n)dn. \end{aligned} \quad (10)$$

This can be simplified as

$$\begin{aligned} &(\rho + \theta)v(m) \\ &= \frac{\theta}{2} \int \left[\begin{array}{l} v(m - r(x(m), y(n))) \\ +v(m - r(x(n), y(m))) \end{array} \right] h(n)dn + \frac{\theta}{2}\lambda(m). \end{aligned} \quad (11)$$

Stationary equilibrium (6)

A stationary equilibrium is a set of functions satisfying the above three sets of conditions.

Construction of a fixed point mapping

A mapping from $V \times H$ to $V \times H$

- Step 1: Use the condition for individual optimization to construct a mapping from $V \times H$ to $X \times Y$
- Step 2: Use (a modified version of) the condition for stationarity of money distribution to construct a mapping from $H \times X \times Y$ to H .
- Step 3: Use the condition for stationarity of the value function to construct a mapping from $H \times X \times Y$ to *symbol* V .

Combining these steps, we obtain the mapping \mathcal{F} from $V \times H$ to $V \times H$.

We suppose that the above mapping has a fixed point and denote it by (v^0, h^0) .

$$(v^0, h^0) = \mathcal{F}(v^0, h^0) \tag{12}$$

Perturbations of (v, h) (1)

We consider one-parameter perturbations (\tilde{v}, \tilde{h}) around the fixed point (v^0, h^0)

where for each ε , $\tilde{v}(\varepsilon, m)$ is a function from M to $[0, \infty)$ and the same for $\tilde{h}(\varepsilon, m)$.

Let $\tilde{v}(0, m) = v^0(m)$ and $\tilde{h}(0, m) = h^0(m)$.

Let, for each ε , (\tilde{v}, \tilde{h}) be defined by

$$(\tilde{v}, \tilde{h}) = \mathcal{F}(\tilde{v}, \tilde{h}). \quad (13)$$

Conjecture (1)

If, for each ε , (\tilde{v}, \tilde{h}) coincides with (\tilde{v}, \tilde{h}) , then, the pair of functions for each ε constitutes a fixed point.

In particular, in such cases, $(\tilde{v}_\varepsilon, \tilde{h}_\varepsilon)$ and $(\tilde{\tilde{v}}_\varepsilon, \tilde{\tilde{h}}_\varepsilon)$ defined as follows must coincide.

$$\tilde{v}_\varepsilon(m) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\tilde{v}(\varepsilon, m) - \tilde{v}(0, m)) \quad (14)$$

$$\tilde{h}_\varepsilon(m) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\tilde{h}(\varepsilon, m) - \tilde{h}(0, m)) \quad (15)$$

and similarly for $\tilde{\tilde{v}}_\varepsilon(m)$ and $\tilde{\tilde{h}}_\varepsilon(m)$.

Conjecture (2)

Such combinations of $(\tilde{v}_\varepsilon, \tilde{h}_\varepsilon)$ and $(\tilde{\tilde{v}}_\varepsilon, \tilde{\tilde{h}}_\varepsilon)$ must be related, under some regularity conditions, by a system of linear integral equations of the following form.

$$\begin{aligned}\tilde{\tilde{v}}_\varepsilon(m) &= \int F^{vv}(m, n)\tilde{v}_\varepsilon(n)dn + \int F^{vh}(m, n)\tilde{h}_\varepsilon(n)dn \\ \tilde{\tilde{h}}_\varepsilon(m) &= \int F^{hv}(m, n)\tilde{v}_\varepsilon(n)dn + \int F^{hh}(m, n)\tilde{h}_\varepsilon(n)dn.\end{aligned}\tag{16}$$

The set of kernels $(F^{vv}, F^{vh}, F^{hv}, F^{hh})$ need to be derived by linearizing the mapping \mathcal{F} at the fixed point (v^0, h^0) .

Application of Fredholm's theorem

In order for the linear mapping defined by (16) to have a nontrivial fixed point, that is, in order for there to exist a set of functions such that $\tilde{v}_\varepsilon(m) = \tilde{v}_\varepsilon(m)$ and $\tilde{h}_\varepsilon(m) = \tilde{h}_\varepsilon(m)$ satisfying (16), there needs to exist a nontrivial set of functions $\omega^v(m)$ and $\omega^h(m)$ defined for all m such that for any set of \tilde{v}_ε , \tilde{h}_ε , \tilde{v}_ε and \tilde{h}_ε satisfying (16), it holds that

$$\begin{aligned} & \int \omega^v(m) \tilde{v}_\varepsilon(m) dm + \int \omega^h(m) \tilde{h}_\varepsilon(m) dm \\ &= \int \omega^v(m) \tilde{v}_\varepsilon(m) dm + \int \omega^v(m) \tilde{v}_\varepsilon(m) dm, \end{aligned} \quad (17)$$

Target

To show that such a pair of functions (ω^v, ω^h) exists.

Conjecture

A pair defined by $\omega^v(m) = h^0(m)$ and $\omega^h(m) = -v^0(m)$ works.

Linearizing the mapping (1)

By linearizing the condition for individual optimization, the condition relating $(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon, \tilde{\lambda}_\varepsilon)$ to $(\tilde{v}_\varepsilon, \tilde{h}_\varepsilon)$ is obtained.

$$\begin{aligned}
 & \int \left[\begin{array}{l} u_{xx}(x^0(m), y^0(n)) \\ +v_m^0(m - r(x^0(m), y^0(n)))[r_x(x^0(m), y^0(n))]^2 \\ +v_m^0(m - r(x^0(m), y^0(n)))r_{xx}(x^0(m), y^0(n)) \end{array} \right] \tilde{x}_\varepsilon(m)h^0(n)dn \\
 & + \int \left[\begin{array}{l} u_{xy}(x^0(m), y^0(n)) \\ +v_m^0(m - r(x^0(m), y^0(n))) \\ \times r_y(x^0(m), y^0(n))r_x(x^0(m), y^0(n)) \\ +v_m^0(m - r(x^0(m), y^0(n)))r_{xy}(x^0(m), y^0(n)) \end{array} \right] \tilde{y}_\varepsilon(n)h^0(n)dn \\
 & - \int \tilde{v}_\varepsilon(m - r(x^0(m), y^0(n)))r_x(x^0(m), y^0(n))h^0(n)dn \\
 & + \int \left[\begin{array}{l} u_x(x^0(m), y^0(n)) \\ -v^0(m - r(x^0(m), y^0(n)))r_x(x^0(m), y^0(n)) \end{array} \right] \tilde{h}_\varepsilon(n)dn \\
 & - \lambda^0(m)R_{xx}(x^0(m))\tilde{x}_\varepsilon(m) - \tilde{\lambda}_\varepsilon(m)R_x(x^0(m)) \\
 & = 0,
 \end{aligned} \tag{18}$$

Linearizing the mapping (2)

$$\begin{aligned}
 & \int \left[\begin{array}{l} -c_{yx}(x^0(n), y^0(m)) \\ +v_m^0(m + r(x^0(n), y^0(m))) \\ \times r_x(x^0(n), y^0(m))r_y(x^0(n), y^0(m)) \\ +v^0(m + r(x^0(n), y^0(m)))r_{yx}(x^0(n), y^0(m)) \end{array} \right] \tilde{x}_\varepsilon(n)h^0(n)dn \\
 & + \int \left[\begin{array}{l} -c_{yy}(x^0(n), y^0(m)) \\ +v_m^0(m + r(x^0(n), y^0(m)))[r_y(x^0(n), y^0(m))]^2 \\ +v^0(m + r(x^0(n), y^0(m)))r_{yy}(x^0(n), y^0(m)) \end{array} \right] \tilde{y}_\varepsilon(m)h^0(n)dn \\
 & + \int \tilde{v}_\varepsilon(m + r(x^0(n), y^0(m)))r_y(x^0(n), y^0(m))h^0(n)dn \\
 & + \int \left[\begin{array}{l} -c_y(x^0(n), y^0(m)) \\ +v^0(m + r(x^0(n), y^0(m)))r_y(x^0(n), y^0(m)) \end{array} \right] \tilde{h}_\varepsilon(n)dn \\
 & = 0.
 \end{aligned} \tag{19}$$

The condition for complementary slackness (8) can effectively be neglected by, for example, introducing a new function z such that $x(m) = z(m)$ if $R(z(m)) \leq m$ and $\lambda(m) = z(m) - R^{-1}(m)$ if $R(z(m)) > m$.

Linearizing the mapping (3)

By linearizing the condition for stationarity of money distribution, the condition relating \tilde{h}_ϵ to $(\tilde{x}_\epsilon, \tilde{y}_\epsilon, \tilde{h}_\epsilon)$ could be derived.

But it seems more interesting to see another property satisfied by $(\tilde{x}_\epsilon, \tilde{y}_\epsilon, \tilde{h}_\epsilon)$ and \tilde{h}_ϵ .

Let the mapping from $H \times X \times Y$ to H be defined as follows.

Construction of the mapping

Define $\tilde{k}(m, n)$ and $\tilde{l}(m, n)$ by $\tilde{k}(m, n) = m - \tilde{r}(m, n)$ and $\tilde{l}(m, n) = n + \tilde{r}(m, n)$ where $\tilde{r}(m, n)$ is understood as $r(\tilde{x}(m), \tilde{y}(n))$.

Let $\tilde{\eta}(k, l)$ denote the density of the matches after which the buyer is found holding k units of money and the seller holding l units of money.

Then

$$\begin{aligned} \tilde{\eta}(\tilde{k}(m, n), \tilde{l}(m, n)) & \begin{vmatrix} 1 - \tilde{r}_m(m, n) & -\tilde{r}_n(m, n) \\ \tilde{r}_m(m, n) & 1 + \tilde{r}_n(m, n) \end{vmatrix} \\ & = \tilde{h}(m)\tilde{h}(n). \end{aligned} \quad (20)$$

We define

$$2\tilde{\tilde{h}}(k) = \int \tilde{\eta}(k, l)dl + \int \tilde{\eta}(l, k)dl. \quad (21)$$

Linearizing the mapping (4)

A property satisfied by $(\tilde{x}_\epsilon, \tilde{y}_\epsilon, \tilde{h}_\epsilon)$ and \tilde{h}_ϵ .

Then, for any function $f(m)$,

$$\begin{aligned} \frac{1}{2} \iint \tilde{h}(m)\tilde{h}(n) \left[\begin{array}{l} f(m - r(\tilde{x}(m), \tilde{y}(n))) \\ + f(n + r(\tilde{x}(m), \tilde{y}(n))) \end{array} \right] dm dn \\ = \int \tilde{h}(m)f(m)dm. \end{aligned} \quad (22)$$

Linearizing the mapping (5)

By linearizing the condition for stationarity of the value function, the condition relating \tilde{v}_ε to $(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon, \tilde{\lambda}_\varepsilon, \tilde{h}_\varepsilon)$ is derived.

$$\begin{aligned} & (\rho + \theta)\tilde{v}_\varepsilon(m) \\ &= \frac{\theta}{2} \int \tilde{v}_\varepsilon(m - r(x^0(m), y^0(n)))h^0(n)dn \\ &= \frac{\theta}{2} \int \tilde{v}_\varepsilon(m + r(x^0(n), y^0(m)))h^0(n)dn \\ &- \frac{\theta}{2} \int v_m^0(m - r(x^0(m), y^0(n))) \left[\begin{array}{l} r_x(x^0(m), y^0(n))\tilde{x}_\varepsilon(m) \\ +r_y(x^0(m), y^0(n))\tilde{y}_\varepsilon(n) \end{array} \right] h^0(n)dn \\ &- + \frac{\theta}{2} \int v_m^0(m + r(x^0(n), y^0(m))) \left[\begin{array}{l} r_x(x^0(n), y^0(m))\tilde{x}_\varepsilon(n) \\ +r_y(x^0(n), y^0(m))\tilde{y}_\varepsilon(m) \end{array} \right] h^0(n)dn \\ &- \frac{\theta}{2} \int v^0(m - r(x^0(m), y^0(n)))\tilde{h}_\varepsilon(n)dn + \frac{\theta}{2}\tilde{\lambda}_\varepsilon(m) \\ &- \frac{\theta}{2} \int v^0(m + r(x^0(n), y^0(m)))\tilde{h}_\varepsilon(n)dn. \end{aligned} \tag{23}$$

Currently...

Verifying whether $\omega^v(m) = h^0(m)$ and $\omega^h(m) = -v^0(m)$, that is, whether

$$\begin{aligned} & \int h^0(m) \left[\int F^{vv}(m, n) \tilde{v}_\varepsilon(n) dn + \int F^{vh}(m, n) \tilde{h}_\varepsilon(n) dn \right] dm \\ & - \int v^0(m) \left[\int F^{hv}(m, n) \tilde{v}_\varepsilon(n) dn + \int F^{hh}(m, n) \tilde{h}_\varepsilon(n) dn \right] dm \\ & = \int h^0(m) \tilde{v}_\varepsilon(m) dm - \int v^0(m) \tilde{h}_\varepsilon(m) dm \end{aligned} \tag{24}$$

where F 's are derived in a manner as described above.

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- Kondo, Jiro. Integral equations. Oxford Univ. Pr. (1992).